

Mathematics as a Bridge Between the Disciplines

Proceedings of MACAS – 2017 Symposium

Edited by
Claus Michelsen
Astrid Beckmann
Viktor Freiman
Uffe Thomas Jankvist

Mathematics
And its
Connections to the
Arts and
Sciences


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MATHEMATICS AS A BRIDGE
BETWEEN THE DISCIPLINES
PROCEEDINGS OF MACAS – 2017 SYMPOSIUM

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INTRODUCTION

The symposium series MACAS, Mathematics and its Connections to the Arts and Sciences, was founded in 2005 by Astrid Beckmann, University of Education Schwäbsich Gmünd, Bharath Sriraman, The University of Montana and Claus Michelsen, University of Southern Denmark as an outcome of the continued collaboration between some participants of Topic Study Group 21 at the 10th International Congress of Mathematics Education (ICME-10), held in Copenhagen in July 2004 (Anaya & Michelsen 2005, 2008). The first MACAS symposium was held in 2005 at the University of Education Schwäbsich Gmünd, Germany (Beckmann, Michelsen & Sriraman 2005). Subsequent MACAS-meetings were held at University of Southern Denmark in Odense, Denmark in 2007 (Sriraman, Michelsen, Beckmann & Freiman 2008), and at University of Moncton, Canada in 2009 (Sriraman & Freiman 2011). For its 10th anniversary in 2015 MACAS turned back to University of Education Schwäbsich Gmünd (Beckmann, Freiman & Michelsen 2016) and in 2017 it returned to Denmark, this time at Danish School of Education, Aarhus University in Copenhagen.

Mathematics is part of almost every aspect of everyday life, and the society consumes a lot of mathematics. Across regions, nations, and continents mathematics plays a central role in educational systems from kindergarten to lifelong learning. Mathematics plays an increasingly important part in many scientific disciplines like the physical, the engineering, the biological sciences, information science, economics, sociology, linguistics and dozens of other disciplines as well, although the way in which mathematics is involved in them varies considerably with the discipline. The vision which the MACAS-initiative is based upon is to develop a holistic approach to education that combines various disciplines in a single curriculum – an approach first suggested by renaissance philosophers. According to this philosophical notion, the aim is to educate students by enabling them to pursue diverse fields of inquiry while at the same time exploring the aesthetic and scientific connections between the arts and science. In view of the challenges of the 21st century, a modern approach to education with a focus on multi- and interdisciplinary is more important than ever. The field of mathematics assumes a key role in this approach as it is connected to all other disciplines and can serve as a bridge between them. This is the approach of MACAS – Mathematics and its Connections to the Arts and Sciences.

The MACAS 2017 symposium took place at Danish School of Education, Aarhus University in Copenhagen 27 - 29 June 2017. It included 42 participants from Canada, China, Denmark, Faroe Islands, Germany, Mexico, Russia, Sweden, Switzerland, Ukraine and United Kingdom interested in connections between

mathematics and the arts and. The following areas were in focus at the symposium:

- Theoretical investigation of the relation between mathematics, arts and science
- Curricular approaches to integrate mathematics and science
- The importance of mathematical modelling and interdisciplinary for studying and learning mathematics
- The importance of arts and humanities for the understanding of the connections between arts, humanities and mathematics in ordinary everyday situations
- Intercultural dimensions of studying mathematics

These proceedings collect papers corresponding to the plenary lectures and presentations given at MACAS 2017 symposium. The proceedings present 19 peer reviewed papers. The papers are very diverse in nature reflecting the fact that impacts of mathematics can spread very in many cases. However, this diversity points at the need for a community-wide effort to rethink the mathematics education at all levels. Ideas, experiences, conceptual frameworks, and theories to connect mathematics education to the arts and sciences need to be improved to meet the challenges and opportunities of the future.

From the symposium's plenary sections, the paper by Annie Savard (Canada) discusses how critical thinking using mathematics might support the decision-making process from an ethnomathematical perspective. Jens Højgaard Jensen (Denmark) shares his reflections about the distinction between theory-derived mathematical models and ad-hoc mathematical models as a way to help ordinary people, not to distinguish between trustworthy and non-trustworthy models, but to distinguish between the different qualities of the evaluation proses behind different sorts of models. A third plenary (without proceedings paper) was given by Paul Ernest (UK) on the topic of "Mathematics, Beauty and Art" in which he addressed the questions what beauty in mathematics is and what dimensions of mathematical beauty that can be distinguished? Provisional answers to these questions were given, and mathematical beauty was illustrated by means of an example from visual art. Since beauty is shared by both mathematics and art, Ernest also asked the question of what parallels, including similarities and differences, that can be drawn between mathematics and art?

Two papers have focus on geometrical objects. Hans Walser (Switzerland) comes across different aspects of equivalence by dissection: Variations on the theorem of Pythagoras, differences between methods and creativity, symmetry, optimizing, rational and irrational rectangles, color and esthetics. The paper of Gao Shuzhu, Chen Weiwei and Zheng Qian (China) explains the volume of a cone by the concept of a centroid.

A group of papers address the connections between mathematics and the subjects of natural sciences. Thomas Højgaard and Jan Sølberg (Denmark) present a two-dimensional model to ensure that students acquire competencies that transcend traditional subjects. The paper by Martin Niss (Denmark) focuses on how the students' ability to perform the mathematization process can be trained by using so-called unformalized physics problems. The paper by Claus Michelsen (Denmark) reports about an in-service teacher program aimed at enabling teachers to implement interdisciplinary instructional sequences in mathematics and biology in their daily classroom practices. Simon Zell (Germany) discusses different approaches for models of interdisciplinary teaching and presents his own model "Mathematics and Science under one roof".

Topics related to technology in mathematics are addressed in the papers by LeBlanc, Freiman and Furlong (Canada) with focus on emerging mathematical connections when students are learning in school makerspaces and students' motivation for learning mathematics when technology-based games are integrated within the classroom.

Several papers address the connections between mathematics and literature, music and arts. Starting out with G. H. Hardy's aesthetic arguments for the value of pure mathematics the paper by Uffe Thomas Jankvist, Helle Rørbech & Jesper Bremholm (Denmark) points out didactic potentials in an interdisciplinary approach to beauty and aesthetics within the context of mid-20th century ways of thinking and understanding mathematics and literature. Irina Golovacheva, Alexandre Stroeve, Mikhail Zhuravlev and Polina de Mauny (Russia) analyze the structure at the artistic space of two world-famous masochistic novellas by mathematical modeling. The paper by Lina Medina Ibarra, Avenilde Romo-Vázquez & Mario Sánchez Aguilar (Mexico) presents an activity centered on an analysis of the story of Jorge Luis Borges "The library of Babel" from a literary as well as from a mathematical point of view. Hans Peter Nutzinger (Germany) shares the idea that music is a way of learning about patterning and thereby about mathematics. The use of terahertz electromagnetic oscillations in art expertise and public art technologies is analyzed in the paper by Darya Yeryomka (Ukraine).

Giftedness, creativity and aesthetic are explored in three papers. Peter Weng and Uffe Jankvist (Denmark) address the problem of many teachers not being

equipped for engaging in dialogue with gifted students and thus not being able to facilitate their mathematical learning in a productive and efficient manner. Lena Lindenskov (Denmark) presents the “Seven keys” model as a theoretical background for combining aesthetic aspect of mathematics research and mathematics learning. In the paper by Lissner Rye Ejersbo (Denmark) three cases are presented to discuss how to make mathematics a creative subject.

Finally, Maria Kirstine Østergaard (Denmark) argues that it is essential to focus on the development of students’ beliefs in mathematics education, particularly about mathematics as a discipline, in order to enhance the students’ apprehension of the role and use of mathematics in the world and to emphasize the interdisciplinary possibilities of mathematics.

The overall success of the MACAS 2017 Symposium was a result of a very productive scientific work magnificently supported by the great enthusiasm, devotion and hospitality of the local organizing team lead by Professor, Dr. Uffe Jankvist promotes for the continuation of the MACAS symposia in the coming years. The 6th one is planned in 2019 in Montreal, Canada.

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MAKING DECISIONS IN A COMPLEX WORLD: TEACHING HOW TO NAVIGATE USING MATHEMATICS

Annie Savard¹

Abstract

We are living in a very complex world and the complexity is increasing by the knowledge needed to make relevant decisions for ourselves and our communities. Community, crisis, economy, identity, health, sustainability and technology are some of the prominent facets of this complexity, that make the world we used to know a different place to be in today. The world is changing so fast, and without strong knowledge and skills, it is hard to navigate it. How can we support our students to make relevant decisions for themselves and their communities? How can we teach them knowledge and skills when the jobs they will have don't exist yet? This plenary session will present how critical thinking using mathematics might support the decision-making process from an ethnomathematical perspective.

Living in a complex world

The world is changing so fast that we are constantly adapting the knowledge we use in daily life. All aspects of modern societies are in fact engaged in an adaptation process: work places and economies, transportation, and personal life. For instance, the workplaces are changing a lot with processes such as automatization and robotization: offices, farms, factories, stores, banks are few examples where computers and robots are used in a way that less people are needed for doing the work. Technologies changed the world of business by proposing new models of making money: start-up, incubators, and crowdfunding are recent examples of these important changes. The use of digital money is changing the local and the global economies by providing an incredible accessibility of goods and services. In many cases, there is no need to be physically present to have access to them. The changes in personal and public transportation vary a lot, but all of them aim to promote efficiency and safety: global positioning system (GPS), backup cameras, and electrical cars are just few examples for personal transportation. Public transportation changed a lot with

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the digital world. It is now easier to use it by looking at online schedules and making digital payments in bus. Those are just few examples, but I am sure you can think of many more.

Our personal life style is also adapting with the technological advances. All our environment is now shaped with new technologies who impacted our interpersonal relationships. The influence of social networks and digital security shapes many aspects of personal life. The change in the medias of communication made it easy to communicate in many different ways. The world is changing in so many directions that it is hard to get a big picture.

The institutions of learning are changing as well. Some of them change faster than others. They offer asynchronous courses online so that participants can be everywhere in the world. For example, learners can stay home and be trained in another country. The online offers for training are appealing for many learners. The accessibility and the quality of certain offers make serious competition to local institutions. On the other hand, the IT tools shaped the process of learning as well. They proposed many ways to interact with content. Some people, wizards of the future, even predict that teachers and professors will not have job in the future! It is relevant to question then, what are the jobs of the future? Are we preparing our students for those jobs? Can we use the same tools used 30 years ago to teach them? Do we educate them to apply knowledge or do we educate them to think? How can we prepare them for the jobs which do not yet exist? In other words, how can we support them to navigate in our complex world? It is enough to focus on knowledge acquisition and application of techniques? Are we teaching them how to use a compass in our GPS world? It is enough to find the North or read a map when navigating today and tomorrow? For example, in the case of a blocked road, finding another way is the only option, and what we really need is to make a good decision as to where to go next.

Making decisions is what we are doing all the time, implicitly and explicitly. Just think about that: when are you making decisions in your daily life? Are you making decisions when you drive? When you teach? When you talk or write? We are making decisions all the time. We are not fully aware of the amount of decisions made in a single day: what to wear, what and when to eat, what to watch, what to do, when to go to bed, where to go, what to say next to students... Some decisions are more important than others. For instance, buying or renting a car or a house, having children or selecting a career are big decisions to make. In the other hand, the everyday decision on what to eat is important given the frequency and the impact of health. Whatever the importance of the decision to

be made, I think that making decision should be done thoughtfully, because its impact at different degree our life and our community. It is easy to say, but what really means and implies making thoughtful decision? What is the role of mathematics in that process?

Making thoughtful decision

The decision-making process can be seen as a competency, because it involves a dynamic (re)-organisation of resources (Legendre, 2004). This (re)-organisation may be interpreted as a mobilization of different resources, that is, knowledge, processes, tools, mental constructions. Mobilizing knowledge is not just transferring knowledge, because knowledge is often transformed and/or expanded according to the situation: "mobilize is not only 'use' or 'apply', it is also adapt, differentiate, integrate, generalize or specify, combine, orchestrate, so conduct *a set of mental complex operations* that, connected to the situations, transform knowledge instead of moving them" (Perrenoud, 2002, p. 46). In the decision-making process, the knowledge and other resources are mobilized accordingly to the situation and the decision to be made.

The starting point of the decision-making process is to recognize that there is a need to make decisions (Halpern, 2003; Paul & Elder, 2001). It is important to know why a decision should be made (Swartz & Perkins, 1990). Making a decision implies that it might impact not only oneself, but also others. In fact, there are individual and social responsibilities (Swartz & Perkins, 1990). Recognizing the need to make a decision also implies knowing the short-term and long-term effects. Usually, the choice made tends to have a short-term effect, which is getting instant gratification or pleasure, (Paul & Elder, 2001). Considering long-term decisions requires a deeper thinking and self-discipline, because is not always possible to predict the long-term effect (Paul & Elder, 2001). Recognizing the need to make a decision allows us to generate and explore different options and then to assess them in order to find the best choice (Swartz & Perkins, 1990). One option might emerge intuitively as the most desirable, but further consideration usually leads to other possibilities. Other considerations include personal feelings, values and knowledge, cognitive and cultural bias, and environmental variables (availability of an object for example) (Halpern, 2003). Assessment of the options is critical in order to select one that can meet the criteria identified by the individual. In fact, assessing the options should reflect the "pros" and "cons" of the them. Therefore, it involves taking into consideration the probabilities, the possible consequences and the possible risks or benefits involved. It comes to choosing the "best" option, but the best for whom? And based on what criteria? Is the "best" option a short or a long term one? Assessing the options involves considering the uncertainty of the decision chosen. Thus, it

is important to be creative in generating the options in order to assess the maximum possible contingencies for self and others' wellbeing. The context of the decision plays an important role here, because it is composed of social and cultural factors, the kind of decision to make (short or long term), technical issues or personal implications (importance of the decision).

The selection of an option is followed by the action to undertake. It is helpful to ask about the actions involved with this option. This leads to the re-assessment of the decision and finding other options in a dynamic and iterative process. Thus, the cycle starts over again by the recognition of the need to decide, followed by the generation and assessment of options, selection of an option, and finally the actions to undertake. At this point, the nature of the decision to be made, the context, or the options could be changed throughout the decision-making process (Halper, 2003). The re-assessment of an option also leads to making a decision and validating the choice afterward. It is possible to look back at the process and see if the option chosen was effectively the best one in this case.

A thoughtful decision is made using critical thinking. Thus, it is not possible to generate, assess and select options without being critical. Asking critical questions reflects a thoughtful thinking. Table 1 present some examples of questions for each step of the decision-making process.

Decision-making process	Critical thinking opportunities
Recognize the need to make a decision	<ul style="list-style-type: none"> • Do I really need that? • Why this decision is important to be made for me? For my family?
Identify the options	<ul style="list-style-type: none"> • Can I think to other options? • What are my bias here?
Assess the options	<ul style="list-style-type: none"> • Are they valid options in this case? • Are the risks worth it?
Selection of an option and actions to undertake	<ul style="list-style-type: none"> • Do the actions to undertakes are feasible? • Are the actions to undertake costly?
Re-assess the decision and find other options	<ul style="list-style-type: none"> • Did I select the best option in this case? • What might be an unexpected outcome with this option?

Make the decision	<ul style="list-style-type: none"> • Is my decision supported by valid arguments? • How can I justify this option to my peers?
Validate the choice made	<ul style="list-style-type: none"> • Did I select the best option in this case? • What I world do next time?

Table 1: Articulation of critical thinking into decision-making process

In fact, critical thinking facilitates the judgement by using criteria (including models), by being self-correcting and by taking into account the environment (Lipman, 2003). Criteria used to make a judgement are the basis of justification and comparison, because they define the reliability and the validity of the judgment. A self-assessment of thought aims to correct our own mistakes when thinking about something. This self-correction of critical thinking should thus prevent from basing a judgment on personal or cultural bias, beliefs or values. At this end, emotions and affectivity should also be taken into consideration, because attitudes influence thinking especially in situations that require judgment or decision making about emotional or emotional issues (Bailin, Case, Coombs, & Daniels, 1999; Yinger, 1980). Taking into account the environment or the context allows one to determine the relevance of the criteria, to relativize the judgment and consider the implications. For instance, it involves considering aspects that can reflect and provide an overview of the circumstances and limitations of an event: exceptional or unforeseen circumstances; contingent or special constraints limits; of an overview; the possibility that the example is atypical; the possibility that meaning is untranslatable from one context to another (Lipman, 2003). This situated thinking is emancipatory, because it leaves room for doubts by giving up common ideas.

As I stated earlier, creating a judgement or a point of view is based on criteria that can be used to justify thoughts. A justification is based on arguments made to support the judgement made. An argument leads to examine the acceptability of reasons given during a justification (Duval, 1992-1993). Arguments are considered valid based on their strength and their relevance. The strength of an argument is the resistance against-one counter-argument and its epistemic value, defined as the degree of certainty or conviction tied to a proposal (Duval, 1991). In this sense, mathematics can be used to generated arguments. In fact, critical thinking might contribute to the learning of mathematics by supporting the thinking process, but also mathematics might contribute to the development of

critical thinking by supporting the judgements made using quantification, measures or data. For instance, examining, reporting and evaluating all aspects of a situation or problem, including collecting, organizing, storing, and analyzing information, support the learning of mathematics. Drawing conclusions from the information and identifying inconsistencies and contradictions in the data are other examples on how critical thinking might support the learning of mathematics (Krulik & Rudnick, 1999).

I consider the decision-making process and critical thinking as two citizenship competencies. A citizenship competency is defined as an individual participation in a critical and responsible way to the social practices valued in the community. This participation has to be done in a democratic way to benefit to the society evolution (Ten Dam & Volman, 2004). In other words, each member of the society has to be able to make choices and know why they made those choices, respect the choices and opinions of others, discuss the choices made, make his own opinion and share it with others (Halpern, 2003; Paul & Elder, 2001; Swartz & Perkins, 1990). Decisions can be made on the individual or collective levels, depending of the needs of the person. Thus, the citizenship competencies are needed by all to use different and sometimes contradictory information to adapt to our changing world. The question is now how to support students to develop citizenship competencies using mathematics to navigate our complex world.

Teaching how to navigate using mathematics

I think studying our society using mathematics is the entry point to develop citizenship competencies. Let me explain how. If we propose to our students to study a phenomenon, we anchor their learning in a sociocultural context. They learn about a phenomenon they are likely to meet in their life (if it is not already part of their culture). This sociocultural context is considered as ethnomathematical, because it involves not only mathematics, but mathematical practices used in a cultural group:

Ethnomathematics is a research program of the way in which cultural group understand, articulate and use the concepts and practices which we describe as mathematical, whether or not the cultural group has a concept of mathematics (Barton, 1996, p. 214).

The cultural group does not need to be exotic to be considered as ethnomathematics: "Moreover, ethnomathematics is found not only in exotic cultures but also in the day-to-day practice of groups within our own culture" (Mukhopadhyay & Greer, 2001, p. 309). I would like to discuss one example of how mathematics, sociocultural phenomena and citizenship are connected.

When I taught in elementary school, I proposed to my Grade-4 students to study some gambling activities. I wanted them to know more about such practices, because I observed their behavior after gambling Pokemon cards. They lost most of the time their cards and they were devastated. They were not able to see the likelihood behind the activity. As an effect, they presented a magical thinking to explain the outcomes (Savard, 2014). They were sure that they would be able to get back their cards because they would be luckier in the next time. This phenomenon has been studied among adults decades ago by Langer (1975). It is called “illusion of control”. The gambler thinks he or she can control the outcomes of a gambling activity. This is a non-sense, because it is impossible to predict with certainty the outcome of a gambling activity, even if some skills might be used in it. The outcome is mainly or totally based on chance and there is an irreversible stake of money, object or action (Arseneault, Ladouceur, & Vitaro, 2001; Chevalier, Deguire, Gupta, & Deverensky, 2003). The stakes are irreversible: playing a board game Monopoly is not gambling, because all the money goes back in the box at the end of the game. This illusion of control might look like having a lucky number or a lucky charm, consulting horoscope, having a specific routine such as talking to a slot machine before or after putting money into it. Usually, gamblers lose money and then want to participate more to win their money back or have more. Gambling can lead to addiction to those kinds of activities. It is hard to recover from an addiction to gambling, and this addiction can have negative consequences on gamblers and their surroundings (Ladouceur, 2000). According to Crites (2003), prevention should be start at elementary school.

In order to study phenomena anchored into the sociocultural context, mathematics can be used to analyse social and political questions (Mukhopadhyay & Greer, 2001) Thus, mathematics is a critical tool for making sense of our physical and social world, because mathematics allows the modelization of a phenomenon in a more formal and abstract way. The learning goes then into a mathematical context, where the model made is used to compute a solution to be interpreted mathematically. Then, the learning goes back to the sociocultural context to study the implications of the results on the phenomenon studied. The study of the implications of the results might also lead to another context, the citizenship context. This context goes beyond sociocultural, which is embedded in it, because this context raises questions about participation of debate in society (Savard, Manuel & Lin, 2014). For instance, when I proposed to my Grade-4 students to study some gambling activities, I wanted them to learn about the mathematics involved in those activities: the probabilities (Savard, 2011). I also wanted them to realize the risks of participating: the low probabilities of wining and the risk of addiction (Savard, 2015). So, I proposed to look at some gambling activities to figure out how they work, i.e. modelize them

using probabilities (Savard, in press). The awareness of the high probabilities to lose might be considered as an implication in the citizenship context, because they questioned their participation.

A teaching experiment on how to navigate using mathematics

Here I am providing an example on how my Grade 4 students made sense of mathematics used in gambling. As I stated earlier on, I wanted them to study some gambling activities in order to reason mathematically about gambling and thus develop a critical thinking stance. My group of students attended a public school in a suburban environment and were coming from different socio-economic backgrounds. All the 27 students (15 boys and 12 girls) were more than happy to participate in this research project, because “they helped their teacher to do research with them”. I video-recorded all the lessons and I analyzed the transcripts. I used pseudonyms to respect their privacy. I looked at their representations of mathematics, their alternative conceptions (Savard, 2014), and the kind of thinking they used: mathematical, critical, probabilistic.

The lesson I designed was titled “Create a scientific experiment with your lucky charms”. This lesson aimed students to explore qualitatively subjective probability and develop critical thinking toward the means to estimate probability to win. A certain Friday afternoon, I asked my students if they have a lucky charm. Without any surprise, most of them answered positively. I asked them to tell me what their lucky charms were: cuddly toys, jewels, little toys, coins. I asked them if their lucky charms worked well and they unanimously agreed. I asked them to bring them the next class, so the next Monday. It was bad timing, I know that. The following Monday morning, many of them had it in class. They presented them briefly. I then reminded them that they told me their lucky charms work well and I wanted to know for sure if it was true. They wanted to use the past information to come up with conclusions: I said that they had to check it now by creating a scientific experiment to test them all. I proposed them to work in team of 6 to create a drawing to test their lucky charms. Not everyone within each group had a lucky charm: some did not have one at all, while others left it at home. I made sure that each team had at least one of them. One student said that if they tested a friend’s lucky charms, it would not tell her if her own lucky charm works. I replied that they had to discuss that problem and come up with some conclusions. In fact, they had to design a method on their own that they would have to explain to other groups of students. A discussion followed the experimentation. I opened the discussion by asking the same questions for each team: who in the team had a lucky charm and what are they, then the methodology used to run the experiment and the conclusions. Each member of the team had an opportunity to talk.

The first team, composed of 5 boys, concluded that their lucky charms work well and give them luck. When I asked about their methodology, they said that they put their names in a tuque (winter hat). They did not put Bertrand's name into it, because he did not have his lucky charm with him. They drew until someone gets 3 times out. Then, they played 4 times and they had a big winner. But because all of them won, their lucky charms work. As a teacher/researcher, I did not expect those strong conclusions. Instead of answering them, I asked other students to present their methodology and their conclusions.

The second team, composed of 5 boys, concluded that their lucky charms work in their head: "If we say my lucky charms will be lucky, I will win, this is what makes luck" (Edouard). I tried to find a counter-example to make them think in other contexts. I wanted them to be critical thinker, so I pressed on them by asking them to think if they work in a drawing. Edouard replied sometimes yes, sometimes no. François said that it was chance or hazard.

- Annie: It is hazard? What do you mean?
- François: Sometimes you can lose, sometimes you can win, because... it is hard to explain...
- Annie: It is hard to explain?
- François: Yes. Even if I have a lucky charm, it might be anything, I can still win. I can still win even if I don't have a lucky charm.
- Annie: You can still will even if you don't have it, that's it?
- Édouard: Some people believe in it a lot. If they don't have it, they will be desperate and they'll say I lose, I do not have my lucky charms.
- Annie: Will they still win?
- Édouard: No, they will lose because they will say that. We have to be positive.
- Annie: So the fact that they will lose, they lose. Even in a draw, or in all situations? When you say they are going to lose, do you speak when they play soccer?

- Édouard: It was him who said that in his head, I lose, I do not have my lucky charms.
- Annie: And this has an effect on the outcomes of the drawing?
- Édouard: Not really in a drawing, but when you play soccer. If you say you will lose, you will lose.

It was an important discussion here, because it showed the limits of the lucky charms. Thus, someone can win a drawing with or without a lucky charm. The lucky charms do not work all the time, and you can win without them.

The third team, composed of 4 girls, concluded that their lucky charms have the same power. When explaining their conclusion, they said that a lucky charm “helps to believe we can do it. It helps us overcome trials.” They explicitly said that they built upon Edouard's idea about the use of lucky charms, here that it has some impact when there are actions involved by the owner of the lucky charm.

The fourth team, composed of 4 boys, concluded that it helps to have a lucky charms, because it helps to focus what numbers will chose. The methodology employed was that the only student who had a lucky charm should guess the number chosen by his teammates. He was not able to find the numbers all the time, but they concluded that it was better than nothing. He said that he received cheers from Fred.

The fifth team, composed of 4 girls, concluded that it was really chance or hazard. The lucky charm cannot always bring luck. To reach that conclusion, they put their name on a piece of paper and draw. They noticed that Amélie used a bigger paper than the others, so that “hers came out more often” (Nina). This is a nice example of critical thinking, where the conclusions are not taken for granted. Furthermore, the critical thinking employed by Nina helped her classmate to make an important distinction between hazard and skills:

Lucky charms encourage. Well, in a drawing, it does not really encourage ... But suppose it's a soccer game, it can encourage to have a lucky charms, it may encourage. A draw is like a little chance.

This distinction is important, because it addresses directly the illusion of control thinking. Thus, it is not possible to control the outcomes of a gambling activity, because this kind of activity is based on chance or hazard.

The sixth team, composed of 4 girls and 1 boy, concluded that lucky charms do not work. Because they only had one in their team, they used it by passing it around to test it.

Among the six teams, the conclusions about the efficiency of lucky charms were different: it works well, it works sometimes, it does not work at all. I then asked if we can trust our experiments and if they are valid. The whole class said: *Yes! Even if we used different processes, we have similar results: It is an object that makes you believe in yourself.* Through the discussion, the students co-constructed the distinction between skills and chance. It is because they were critical thinkers toward the limits of the effect of the lucky charms that made them conclude that. I wanted to highlight other methodological limits of the experiments. Thus, I asked at the whole class about the methodology employed by the first team:

Annie: Here in this team, they were 5 students. They experimented only with 4 names, Bertrand has not experienced because he did not have his lucky charms. Does it change the experiment?

Simone: We cannot know if it brings luck or not. Because if everyone who participated had a lucky charms, you cannot know if it's luck or not.

Annie: So is what should have been ...

Simone: Well, it should have had Bertrand participate, because we would have known if lucky charms could ... It was mandatory to be there, it would have left less chances to others.

In this case, Simone used mathematical arguments to justify why their conclusions are not valid. She situated herself in a mathematical context to study a phenomenon from the sociocultural context: the effects of lucky charms on gambling activities. The need to modelize the situation contributed to develop critical thinking toward the conclusions of the first team. In this sense, mathematics was used to develop critical thinking, and critical thinking created the need to provide mathematical arguments. Discussing the methodology was important to develop critical thinking, because it brought students to revise their judgement about lucky charms. Their final conclusion was that lucky charms helps increase their confidence, especially when they do something. But it does not mean that they will win a drawing for sure while having a lucky charms. Through this discussion, they used probabilistic vocabulary such as possible, impossible, certainty to make sense of the uncertainty present in the gambling activities. They made the distinction between chance and skills. This distinction is important to make in order to avoid developing or to change their mind about

the illusion of control over gambling activities. It might help them to make a decision to participate or not to a gambling activity. It might also help to them to be able to justify their decision.

In this lesson, teaching mathematics alone was not possible without using critical thinking. Thus, students develop mathematical and critical thinking practices they can use in other context, to make decision for instance. Learning how to make mathematical arguments in order to think critically about sociocultural phenomena is the way to navigate our complex world.

Concluding remarks

To navigate our complex world, we need critical thinking to guide us like a GPS. This critical thinking helps us to assess the information provided in different contexts. Teaching mathematical knowledge alone is not enough, because we need critical thinking to develop mathematical knowledge and we need mathematics to develop critical thinking. Navigating our world means being able to quantify, measure, estimate, classify, comparing, finding patterns, conjecture, justify, prove and generalize *within* critical thinking and *when using* critical thinking. I think that it is not possible to make a thoughtful decision without using mathematics (even qualitatively) and critical thinking. Thus, teaching mathematics should be done in interaction with critical thinking along with decision making process. We should provide tasks that support the development of the two citizenship competencies. Critical thinking and decision making can also be developed into the mathematical context, so that there is no excuse to not explicitly support students to develop them. All citizens should be able to participate in their society in an informed manner. As educators, we have to step in now before someone asks us to step out in the future.

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MATHEMATICAL MODELLING – HIDING OR GUIDING?

Jens Højgaard Jensen¹

Abstract

The increasing scientific management of technology and society, supported by increasingly powerful information technologies (IT), has lead, and leads to increasingly widespread use of mathematical models. This development gives rise to a democratic problem: How can ordinary people judge the conclusions delivered by mathematical models? Are the conclusions to be believed, since “mathematics do-not lye”? Or is it better to lean on the saying: “there are lies, damned lies and statistics”?

In the paper, I will illustrate a crude distinction between mathematical models derived from theories, and ad hoc mathematical models without reference to more global theories.

The distinction cannot be used to evaluate, whether specific models are hiding or guiding. Theory-derived models may be too idealized to be trusted. And ad hoc models may be trustworthy due to their richness of input data.

The value of the distinction is that it makes it clear that some mathematical models, the theory-derived models, besides of the possibility of evaluating them by comparing with empirical data, also can be evaluated by theoretical considerations. Evaluating ad hoc models are, in contrast, restricted to be done by empirical control only.

Thus, the distinction between theory-derived models and ad-hoc models may help ordinary people, not to distinguish between trustworthy and non-trustworthy models but to distinguish between the different qualities of the evaluation processes behind different sorts of models.

Introduction

This paper is not based on a concluded systematic empirical research. It is based on 40 years of unsystematically experiences with, and interests in, its subject. It may be called a philosophical essay.

In 1980 my interest in the problem, “Mathematical models – hiding or guiding?”, lead to my guidance of a project group at Roskilde University on fourth semester,

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working half of their time on the problem “Matematiske modeller og videnskabsteoretiske problemer ved modeldannelse.” (“Mathematical models and epistemological problems when modelling.”) The group compared a model of fish populations, a hydrological model and a model of the geostationary satellite orbit around the earth epistemologically. They (we) concluded that the qualities of the evaluation processes behind the three models were very different. The theory-derived satellite model, besides of the possibility of evaluating it by comparing with empirical data, could also be evaluated by theoretical considerations. In contrast the ad hoc fish population model was only open for empirical evaluation without having a theory to be derived from. The satellite model and the fish population model represented the extremes in a span from theory-derived models to ad hoc models, whereas the studied hydrological model was a mixture of empirical grounded ad hoc assumptions and theoretical derivations.²

Frequently mathematical models are of the mixed type like the hydrological model. Nevertheless, more clear examples of theory-derived models, like the satellite model, and, in contrast to that, clear examples of ad hoc models like the fishing population model, may be needed as a pedagogical remedy to help the general public to judge whether apparent mathematical models are hiding or guiding. However, it is not sufficient to meet the needs of most people that the examples, like the satellite model and the fish population model, are clear. Besides being clear the examples should also be simpler than those models, if I, as I wanted, should contribute more to oblige mathematical models being a democratic problem: How can the general public judge the conclusions delivered by mathematical models? Are the conclusions to be believed, since “mathematics do-not lie”? Alternatively, is it better to lean on the saying: “There are lies, damned lies, and statistics?”. Therefore, for 10 years I was searching simultaneously clear and simple examples illustrating the points already reached in 1980. The paper here is mainly the result of this search: As simple a presentation as possible of the reflections about “mathematical modelling – hiding or guiding?”. It took half a year to develop the reflections and 10 years to find simple examples communicating the reflections.³

From 1987 to 2008 I and an IMFUFA colleague of mine, Bernhelm Boss-Bavnbek, organized 20 all day seminars at IMFUFA concerning the different use of mathematical models in different subjects and areas, e.g., statistics, applied

² The project report of the students (in Danish) is included in *IMFUFA tekst* no. 26 (1980), Roskilde: Roskilde University. Also included is a more focused journal article (in Danish) by me.

³ The second article (in Danish) presenting the same points with revised examples is included in *IMFUFA tekst* no. 199 (1990), pp. 49-59

mathematics, physics, chemistry, biology, geography, computer science, economy, air pollution, meteorology, bridge building, climate, macroeconomics forecasting, nuclear power risks. This row of seminars gave me a lot of empirical experiences about the use of mathematical models. However, my points from 1980 were not affected. In what follows, I, for pedagogical reasons, present the points schematically by 6 theorems and 3 conclusions.

However, before presenting theorem 1, I want to remind you that the examples in my presentation are pedagogically chosen because they are simple and clear at the same time, not because of their democratic relevance. Hopefully, my discussion can help judging more important, though not simple, examples of mathematical modelling, like:

- The GPS-system
- Climate models
- Macroeconomic forecasting

I will return to and comment on these three examples of mathematical modelling in society in the end of the paper.

Theorem 1: Mathematical models can-not be refused

Mathematics do-not lie. Look at the crane in figure 1.

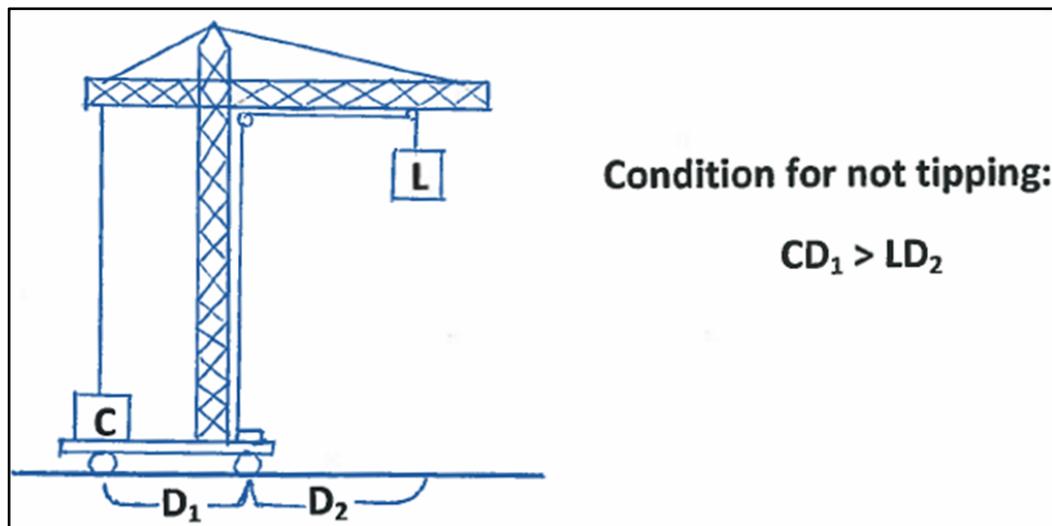


Figure 1: Crane width load L, counter weight C and distances from front wheels, D₂ and D₁, to L and C, respectively.

A crude estimate of the condition ensuring that the crane does not tip around the front wheels is $CD_1 > LD_2$. According to Newtonian mechanics the torque around the front wheels of the crane from the counter weight, have to be larger than the torque around the front wheels from the load. The estimate is crude because it assumes the crane construction to have infinitely small mass compared to L and

D. However, applying the same Newtonian mechanical torque rules as used to make the crude condition, a construction engineer can include the mass distribution of the crane in an almost exact calculation. The point is that the condition for not tipping can be found using mathematics in the framework of Newtonian mechanics. Therefore, both the construction workers on and around the crane, wanting to minimize risks of accidents, and the contractors, wanting to save money, can trust the calculation in spite of conflicting interests. The risks of being exposed when twisting the calculations, is big. Thus, the example proves that mathematical modelling is guiding and do-not lie.

Theorem 2: Mathematical models can-not be trusted

There are lies, damned lies, and statistics. If we want to estimate the risk for crane crashing for all kinds of causes, we have a situation different from calculating the risk for tipping. I believe the insurance companies must have mathematical models helping them to settle insurance prices for crane crashes, where the crash may be due to metal faults in the crane construction, unusual storms, human bypass of the regulations ensuring not tipping, and many other reasons. However, these models are not to be trusted like the tipping model. How can we know that all kinds of causes have been taken into account in the models? How are probability estimates for different thinkable events derived? Judging crash risks for all kinds of causes the construction workers and the contractors shall be aware of their opposite interests and not believe too firmly in conclusions from mathematical modelling with a lot of build in flexibility. This, together with theorem 1, leads us to conclusion 1.

Conclusion 1: Since, 1) mathematical models are gaining increasing importance, and 2) neither blind scepticism nor blind confidence suffice, insight in the diversity of mathematical models is needed

This of course raises the question about the sort of insight needed and how to establish it. However, before coming to that, we turn to a main obstacle for getting the insight in the diversity of mathematical models.

Theorem 3: Mathematical models look alike

For the sake of convincing, you look at the following, not models, but equations:

$$(a+b)(a-b)=a^2-b^2$$

$$c^2=a^2+b^2$$

$$E=mc^2$$

$$F=-kx$$

$$R=PC$$

For ordinary people these five equations look very much alike. They are all letter expressions equating the two sides of the equality sign. However, the similarity of the five equations is superficial. In the first equation a and b represents arbitrary numbers. The equation is an algebraic identity, a consequence of how we operate algebraically, and as such generally valid. The second equation is the Pythagorean Theorem valid for right-angled triangles in a plane. But not for, e.g., right-angled triangles on a sphere. The third equation, due to Einstein, demonstrates the equivalence of mass and energy and is believed to be universally true. The fourth equation is Hook's law expressing how a spring elongates due to a force on it. This is a first order approximation for small elongations and not at all a universal law. Finally the fifth equation belongs to risk analysis. It expresses the risk of an event as the probability of the event times its consequence. The equation is disputed and is to be understood as a definition of risk.

The five equations deal with very different realities although they look very much alike for ordinary people. The same is true for mathematical models. The ways they relate to different realities are very different, although it is overlooked, not only by ordinary people, but also by many professionals. Which lead us to theorem 4.

Theorem 4: In the school, in the disciplines and in the society there exists a cover up of the differences in characters and functions of different mathematical models

As a school related example I will mention how Ohm's law and Coulomb's law were presented in my high school lessons in physics. Ohm's law tells that the voltage difference, V , between the ends of a conductor and the current in it, I , is proportional. That is: $V=RI$, where the proportionality constant, R , is the resistance of the conductor. Coulomb's law tells that the force, F , between two charges, q_1 and q_2 , are equal to a proportionality constant, k , times q_1q_2 , divided by r^2 , r being the distance between the two charges. The two laws are sketched in figure 2.

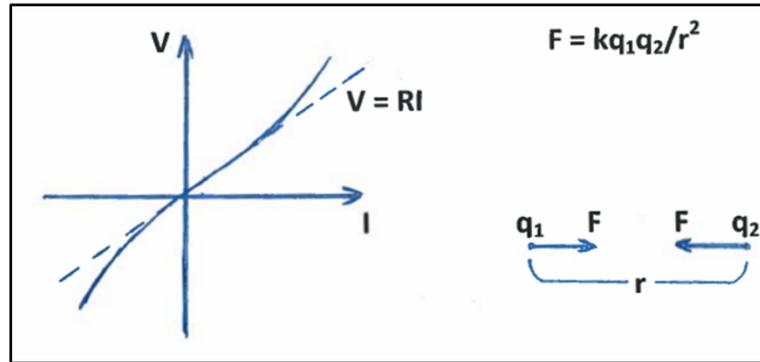


Figure 2: Ohm's Law and Coulomb's law. V is voltage difference, R resistance, I current, F force, k proportionality constant, q_1 and q_2 charges, and r is distance between q_1 and q_2 .

My point here is that the two laws were presented as having the same quality. They were both important equations in electricity theory useful in problem solving. However, in figure 2 it is illustrated that Ohm's law is nothing but a first order approximation. Since I is zero when V is zero and I change sign when V changes sign the function $V(I)$ must go through $(0,0)$ and be symmetric around $(0,0)$. And such a function can in most physical cases be approximated by a straight line not too far from $(0,0)$. Ohm's law is a descriptive ad hoc law like the above-mentioned Hook's law. We should not be surprised experiencing deviations at large values of V and I . Counter wise, we should be very surprised finding proven deviations from Coulomb's law. Coulomb's law is interwoven with Maxwell's basic equations for electrodynamics as a whole. The power 2 of r is, e.g., related to space having 3 dimensions. Thus, substituting r^2 with $r^{1.99999}$ would imply that the whole of electrodynamic theory should be fundamentally revised. However, as mentioned the different epistemological quality of the two laws were not pointed out in the school.

As for a discipline related example, consider figure 3.

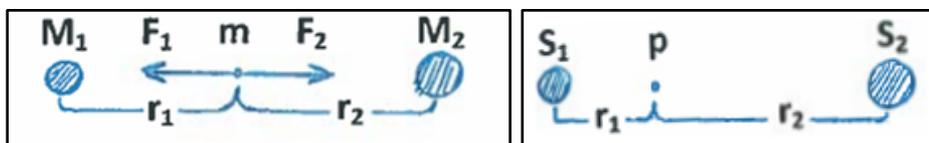


Figure 3: Astronomy and Urban Geography. M_1 is mass of globe 1, M_2 mass of globe 2, m mass of meteor, F_1 force on m from M_1 , F_2 force on m from M_2 , r_1 distance between M_1 and m , r_2 distance between M_2 and m , S_1 size of town 1, S_2 size of town 2, p a person, r_1 distance from S_1 to p , and r_2 distance from S_2 to p .

A part of astronomy is studying the motion of globes due to their gravitational interactions. In the figure, a very simple case is illustrated: A meteor is at rest between two globes. We can then ask: To which side does the meteor with mass

m fall? From Newton's gravitational law we know the gravitational attraction force, F_1 , on m from globe 1, with mass M_1 and distance r_1 from m to M_1 , to be $F_1 = GmM_1/r_1^2$, where G is the universal gravitational constant. Similarly the gravitational force, F_2 , on m from globe 2 is given by $F_2 = GmM_2/r_2^2$. Thus, the answer to our problem is: If $M_1/r_1^2 > M_2/r_2^2$, implying $F_1 > F_2$, the meteor falls towards globe 1. If $M_1/r_1^2 < M_2/r_2^2$, implying $F_1 < F_2$, the meteor falls towards globe 2.

A part of urban geography is studying persons buying behaviour. In the figure, a very simple case is illustrated: A person p lives in between town 1 with size S_1 at distance r_1 and town 2 with size S_2 at distances r_2 . We can then ask: In which town does the person p buy goods? In urban geography, the first guess, according to the so called gravitational law of urban geography, would be: If $S_1/r_1^2 > S_2/r_2^2$ the person buys goods in town 1. If $S_1/r_1^2 < S_2/r_2^2$ the person buys goods in town 2. However, the reference to Newton's law of gravitation is covering up the differences between mechanical astronomy and urban geography. In urban geography it is debated what should be used as a measure of the size of a town. Is, e.g., the number of inhabitants, the number of shopping square meters or the number of parking places the relevant measure for S in the law? In Newton's gravitational law, M is without discussion a mass. In urban geography, the power of r is empirically adjusted from 2 to, e.g., 1.9 or 1.7 for different localities. In Newton's gravitational law, as in Coulomb's law, the power 2 is, via the rest of Newtonian mechanics, related to space having 3 dimensions, and thus not adjustable. Concentrating empirical data in the form of a formula as done in urban geography may be a smart economical idea with many advantages. But it does not resemble Newton's gravitational law. The two examples lead us to conclusion 2.

Conclusion 2: A DISTINCTION between THEORY DERIVED mathematical models and AD HOC mathematical models should be presented in the educational system

The increasing scientific management of technology and society, supported by increasingly powerful information technologies, has led, and leads to increasingly widespread use of mathematical models. Since neither blind scepticism nor blind confidence suffice, the general public needs a cognitive compass helping judging the diversity of the mathematical models. Moreover, since there are limits for the engagement of the single individual, a foundation build on well-chosen examples should be established in the educational system. Furthermore, the apparent similarity of the mathematical models for most people, instead of covering up their differences in characters and functions, in the school, in the disciplines and in the society, as a beginning, a crude

distinction, between *theory-derived mathematical models* like Coulomb's law and the astronomical model on the one side, and *ad hoc mathematical models* like Ohm's law and the urban geography model on the other side, should be presented in the educational system.

With theory-derived mathematical models is meant models interwoven with or derived, as a special application, from a broader mathematically formulated theory. They are deductive in nature. In contrast, ad hoc mathematical models inductively summarize an existing empirical reality in a compact mathematical language without reference to more global theories. The empirical justification of an ad hoc model comes from the apparent context, whereas the theory-derived model gets its empirical justification from the broader empirical scene of earlier model applications of the theory, having consolidated the theory.

Is the distinction unnecessary crude? Do most mathematical models perhaps lie between the two extremes? Yes, but in order to orientate yourself you have to know the extremes. Do I (as physicist) deprive everything besides physics theoretical status? No, but it is correct that the development of physics for good and bad in particular has been interwoven with that of mathematics. Other theories are not in their outset formulated in mathematics. Apart from geometry, combinatorics, and population genetics epistemological character of mathematical models depends on their reference or not to mathematical formulated theory.

However, the distinction cannot be used to evaluate, whether specific models are hiding or guiding. When modelling you always look apart from many aspects making modelling both possible and simple. Thus, theory-derived models may be too idealized to be trusted. Moreover, ad hoc models may be trustworthy due to richness of input data approaching a detailed map of the phenomenon to be modelled. So, what is the usefulness of the distinction?

Theorem 5: Theory-derived mathematical models can be subject to BOTH EMPIRICAL AND THEORETICAL CONTROL, ad hoc mathematical models can ONLY be CONTROLLED EMPIRICALLY

The value of the distinction is that it makes it clear, that some mathematical models, the theory-derived models, besides of the possibility of evaluating them by comparing with empirical data, also can be evaluated by theoretical considerations. Evaluating ad hoc models is, in contrast, restricted to be done by empirical control only.

Figure 4 gives an illustration of what is meant by theoretical control.

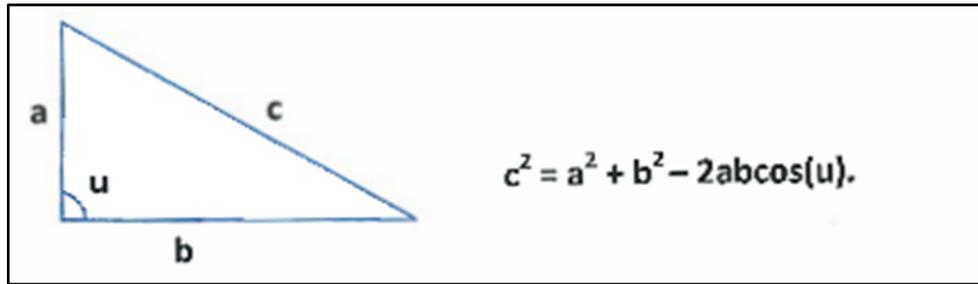


Figure 4: Theoretical control of consequences of deviation from a right angle. a , b and c are sides in an almost right angled triangle. u is the almost right angle.

Let us imagine that the almost right-angled triangle of figure 4 is part of a construction model where it is considered right angled, and thus $c^2 = a^2 + b^2$. Hereby we introduce a failure in our construction calculations since no triangles are exactly right angled in practice. If we only had access to the mathematical model $c^2 = a^2 + b^2$ we were restricted to make loose estimates of the consequences of u varying between, e.g., 89° and 91° . But, since the mathematical model “right angled triangle” is a special case of the broader theory of all sorts of triangles, the consequences of the variations of u for the construction calculations can be estimated precisely using the more general formula for triangles: $c^2 = a^2 + b^2 - 2ab \cos u$. Perhaps with the result that the idealisation $c^2 = a^2 + b^2$ was justified considering the purpose of the model.

We had another example of possibility of theoretical control in the crane example. How small does the weight of the crane construction need to be, compared to the load and the contra weight, in order to justify not considering it when judging the risk of crane tipping? Although more elaborately, this can be calculated using the same torque rules from Newtonian mechanics as used deriving the idealised model result, $CD_1 > LD_2$ (see figure 1). However, if we ask about the risk for a crash of the crane for any cause, no theoretical framework exists to help us judge how we can trust the necessary simplifications made, making a mathematical model estimating the risk. Such a model has to be controlled by confronting it with accident statistics. That is by empirical control. The tipping problem can be placed inside the framework of a mathematical formulated theory in a well-defined manner. Therefore, we are aware of what we ignore in order to make our model simple. Moreover, the induced failures due to the simplifications can be judged currently by the theory. In the crash problem, it is not a question of judging conscious looking apart, the question is rather what we have overlooked. Only empirical data can help us answer that; often with the answer that we have overlooked something, without knowing what.

As mentioned, the distinction between theory-derived mathematical models and ad hoc mathematical models cannot be used to distinguish between trustworthy and non-trustworthy models. The distinction only implies different conditions

for evaluating how trustworthy the models are. Considering this and the abstractness of the distinction one could argue that it is a tame message not worthwhile communicating in the educational system for democratic reasons? But, a closer look makes it perhaps not look that tame.

Theorem 6: Theory-derived mathematical models can be criticised in public by independent experts. In order to criticise ad hoc mathematical models accession to data is needed

The trustworthiness of theory-derived mathematical models versus ad hoc mathematical models cannot be judged just epistemologically. However, the different terms for controlling the two kinds of models, nevertheless often gives sociological reasons for a higher believe in the theory-derived models than in the ad hoc models.

Empirical control of ad hoc mathematical models implies access to data. And this access can be monopolised by owners of the data. Mathematical models in combination with IT most often operate with huge amounts of data. Thus, the collection of data can be expensive, making it hard to carry out empirical control by outsiders.

Theoretical control of theory-derived mathematical models demands insight in theory. This insight is most often reserved for experts with special educations. But among them counter-experts may exist. Moreover, their willingness to deliver theoretical control of a model depends perhaps more on commitment than on money.

Thus, after all, in general, theory-derived mathematical models are more open to public critic and discussion than ad hoc mathematical models. The higher possibility of critic and control from outside experts makes it more risky to tinker with theory-derived mathematical models than with ad hoc mathematical models. Thus, the construction worker have good reasons to believe more firmly on the calculation of the condition for a crane not tipping than on the calculations of crash risks at all.

Besides the final conclusion 3, I hereby have ended the attempt of presenting my points on the hiding or guiding problem of mathematical models using pedagogically chosen examples. As mentioned in the introduction I will here in the end of the paper, before presenting conclusion 3, use the developed concepts to compare my validation of the three, less pedagogical and more societal important mathematical models, mentioned in the introduction.

The GPS-system, Climate models, and Macroeconomic forecasting

I am not an expert in neither the Global Positioning System, or climate models, or macroeconomic forecasting. I have my information about these models from, e.g., Wikipedia. Nevertheless, due to many years of experiences with different kinds of mathematical models and the concepts presented above, I dare judge large qualitative differences between the three models.

The GPS-system operates by triangulating between a receiver and satellites orbiting around 20000 km above the surface of the earth. In order to make the triangulations you must calculate the motion of the satellites using Newtonian mechanics. But you also have to use both special and general theory of relativity! We need to know the positions of the satellites when they send their radio signal to us. However, judged from our position the clock in a satellite is not running with the same speed as our clock. The special theory of relativity tells us that the satellite clock runs slower than ours does because the satellite moves relative to us. The general theory of relativity tells us that the satellite clock runs faster than ours does because the gravitational field from the earth is weaker at the satellite than on the surface of the earth. The combined effect is that we experience a clock in the satellite identical to the clock in our hand running faster. Therefore, the clocks in the satellites are technically adjusted to run slower in order to show our time. Without that adjustment, the GPS-system would not function.

The GPS-system is put under empirical control of all of us using it. It works with astonishing accuracy. But how was it developed to work that precisely. By trial and error? No, without being informed about details, I am convinced that the developing process has been steered by a lot of theoretical control of the mathematical models involved. It is hard to imagine the development of the GPS-system without the theory-derived mathematical models involved, and their current adjustments guided by theoretical control.

Climate models are not throughout theory-derived like the mathematical models of the GPS-system. Parts of them are theory-derived. But they also include more descriptive ad hoc parts. The greenhouse effect due to CO₂ in the atmosphere is theoretically understood for more than hundred years. The selective absorption of electromagnetic radiation of different wavelengths of the CO₂ molecule could be derived theoretically. And the essential thermodynamically consequences of that for the temperature on the earth were foreseen theoretically. However, the temperature variation on the earth caused by fossil fuel burning during the last hundred years is not equally easy to derive theoretically. Besides being influenced by the greenhouse effect the temperature of the atmosphere is, e.g., also influenced by the stirring in the oceans. The heat capacity of the atmosphere equals only the heat capacity of 10 meter of the 4000 km deep oceans. Therefore, there is room for much heat exchange between the atmosphere and the oceans

depending on the not very well understood stirring in the oceans. This heat exchange has to be taken into account more ad hoc like, making forecasting of the temperature on earth rather uncertain.

The theory-derived understanding of the greenhouse mechanisms gives me reason to believe in the order of magnitude of the forecasted temperature increase from climate models. But, the heavy debate about the issue, contrary to a not existing public debate about the models behind the GPS-system, does not surprise me. The debate is of course due to the climate problem being a severe problem with a lot of interests involved in it. But, it is also because the climate models are a mixture of theory-derived models and ad hoc models.

The macroeconomic forecasting models are also a mixture of theory-derived models and ad hoc models. There are many model assumptions in them like using Ohm's law or as in urban geography. And there are many problems connected to the empirical estimation of parameters in these ad hoc assumptions. In these aspects they resemble the climate models. However, they differ from the climate models and the models of the GPS-system, their theory-derived parts being based on a conceptually different kind of theory. In order to compare the macroeconomic forecasting models with the GPS-models and the climate models, we need to distinguish between different conceptual uses of the word "theory", besides distinguishing between theory-derived mathematical models and ad hoc mathematical models.

Theory as believed reality and theory as assumption

Most physicists have a sort of platonic view of the world. For us, a theory is even more real than facts. Empirically registered phenomenological facts may overshadow the essential laws underneath the phenomenological facts. Thus, we believe Newtonian mechanics, special theory of relativity and general theory of relativity to be real. The GPS-system is functioning well because these theories are real.

This does not mean that we believe that the theories are ontologically true, that is, universal and ever-lasting. How *ontological truth* may be established has been discussed by different philosophers with different views, e.g., the logical positivists, Karl Popper and Thomas Kuhn. However, what physicists primarily are interested in when making theory-derived mathematical models is the *pragmatic truth* of theories, that is, their trustworthiness inside specified limits. Have the theories of relativity falsified Newtonian mechanics? Ontologically speaking, yes, since it is possible to formulate problems inside Newtonian mechanics that cannot be formulated inside the theories of relativity, and vice versa. But, pragmatically speaking, no. On the contrary, the theories of relativity

have strengthened the truth of Newtonian mechanics by specifying its operational domain. Thus, physics students at universities are taught to shift between and combine the paradigms of Newtonian mechanics and the special theory of relativity depending on the problems to be solved. Unfortunately, Kuhn's characterisation of Newtonian mechanics and the special theory of relativity, as not comparable ontological paradigms, has fostered the misunderstanding outside physics that physicists may believe in different pragmatic realities. Thus, the public and many economists are misguided, by the interests of the philosophers in ontological matters, to underestimate how huge the epistemological differences between the macroeconomic forecasting models and the theory-derived models in the GPS-system are.

Like the climate models, the macroeconomic forecasting models are composed of a mixture of theory-derived mathematical models and ad hoc mathematical models, the ad hoc models often being weak because of too few empirical data to control and adjust the models. Besides, in economy the word theory is used differently from the use of it in connection with climate models and the GPS-system. Economists use the word the way it is used by pure mathematicians, not by physicists. By a theory is meant an assumption and the logical deductions that can be made from it. Plane geometry is an example of a mathematical theory. Assuming the earth flat, the theory gives results in exact concordance with the theory when developing maps. But of course the theory cannot be used to navigate over large distances on the earth surface, since the assumption made when choosing the theory is wrong. Likewise, macroeconomic forecasting mathematical models are deduced from not verified theoretical assumptions. They may be deduced from, e. g., more neoliberal assumptions or more Keynesian assumptions. In physics, a free choice of theory does not exist. The macroeconomic forecasting models are more prescribing than describing. They prescribe economical actions realizing political goals hidden in their theories. The value of theoretical control in economic models is that it controls their internal consistency. It does not control their trustworthiness. If theoretical control shall ensure trustworthy descriptions of reality, given by a theory-derived mathematical model, the theory have to be believed as a reality due to many and diverse empirical verifications of deductions from the theory. And the empirical verifications of the macroeconomic forecasting mathematical models are few and weak. Competing macroeconomic forecasting models may be a useful qualitative way of ordering and clarifying discussions of the many mutual interactions of factors in the economy. But I do not believe in them as quantitative descriptions of the economic reality.

As mentioned, I am not an expert in neither the GPS-system, or climate models, or macroeconomic forecasting. Besides being a physicist, I have had amateur

experiences with a variety of mathematical models and reflections that are more general on mathematical models in society. This is my background for daring the above judgements comparing the GPS-system, climate models and macroeconomic forecasting. Likewise, it is not necessary for the general public to know details in specific mathematical models in order to have some judgement about their validity and the way the validity is controlled. It is more important being informed about a variety of different simple cases to compare with. This leads me to the final conclusion:

Conclusion 3: 1) It is democratically needed that the educational system supports the public judgement of the diverse character of mathematical models. 2) General philosophical considerations and/or analysis of interests are not sufficient. Comparative, interdisciplinary experiences with mathematical models have to be offered

SEMI-REGULAR FIGURES BETWEEN BEAUTY AND REGULARITY

Hans Walser¹

Abstract

Cutting away a rhombus from a regular pentagon, the leftover will be a semi-regular pentagon. Using this semi-regular pentagon as tile, we can build various ornaments, tilings, crystallographic patterns, and spirals. We will find also a semi-regular not convex dodecahedron. The regular and the semi-regular dodecahedron together can fill the space without gaps or overlapping.

The semi-regular pentagon

The regular pentagon is a bad guy.

We can't use it for a tiling of the plane. There occurs a gap of 36° (Fig. 1c) (Grünbaum and Shephard 1987, Frontispiece).

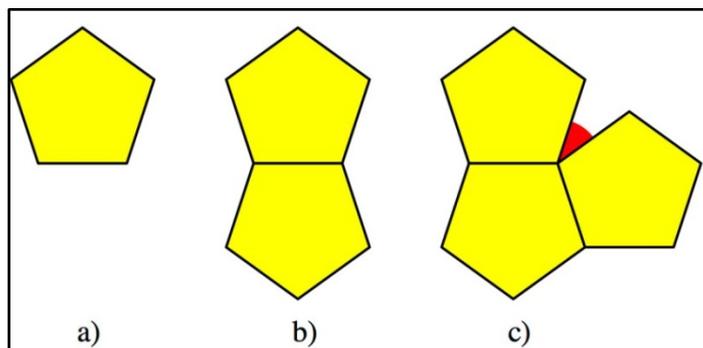
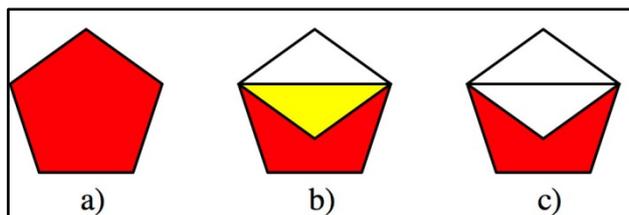


Fig. 1: No tiling with regular pentagons

Now we modify the regular pentagon (Fig. 2a). We flip down a vertex (Fig. 2b). The remaining part will be our semi-regular pentagon (Fig. 2c). The semi-regular pentagon has equal sides, but not equal angles.



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Fig. 2: The semi-regular pentagon

Though our semi-regular pentagon has some defects compared to the regular pentagon, it has also some advantages.

Combination with the regular pentagon

We find a tiling with a combination of regular and semi-regular pentagons (Fig. 3a).

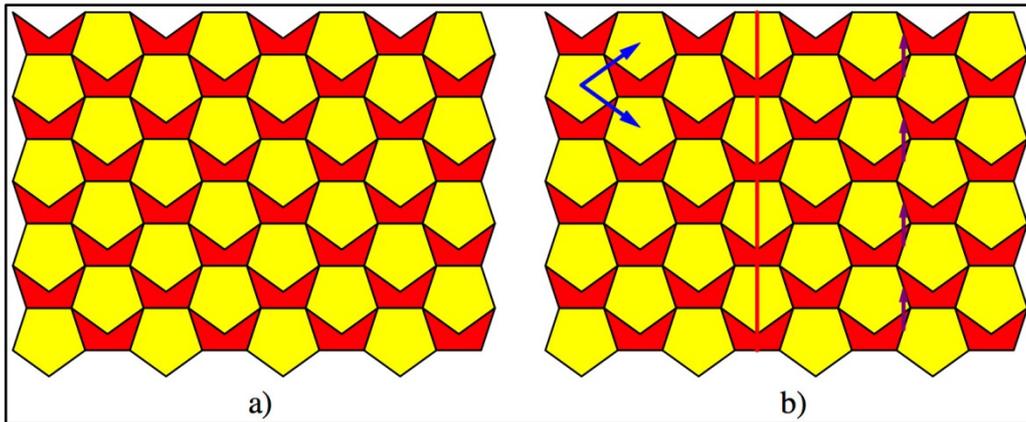


Fig. 3: Tiling with pentagons and semi-regular pentagons

The tiling contains translational symmetries, indicated by blue arrows in the figure 3b, axial symmetries (red) and glide-reflection symmetries (purple).

The figure 4 depicts an example with a fivefold rotational symmetry.

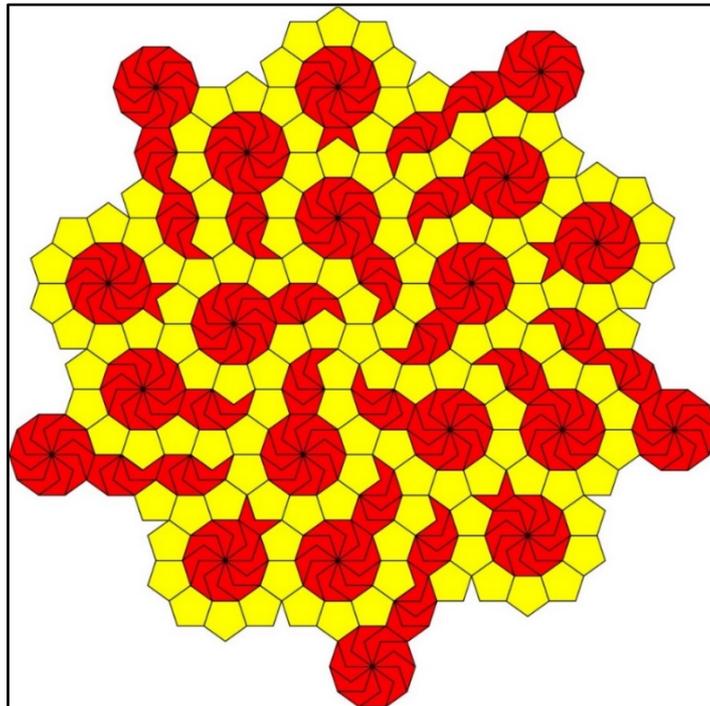


Fig. 4: Fivefold rotational symmetry

Frieze patterns and tilings

Nevertheless we may build frieze patterns and tilings just out of our semi-regular pentagon.

The figure 5 depicts a frieze pattern. With respect to the colors it has translational symmetry only. Without respect to the colors we have additionally a glide-reflection symmetry.

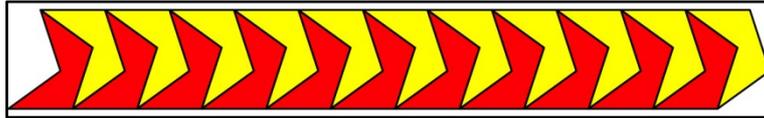


Fig. 5: Frieze pattern

The tiling of the figure 6a is just an overlapping of the frieze pattern of the figure 5. In this figure we may get the victim of an optical illusion. Are the horizontal lines parallel?

The figure 6b depicts a more elegant tiling.

What's the difference between the figures 6c and 6d?

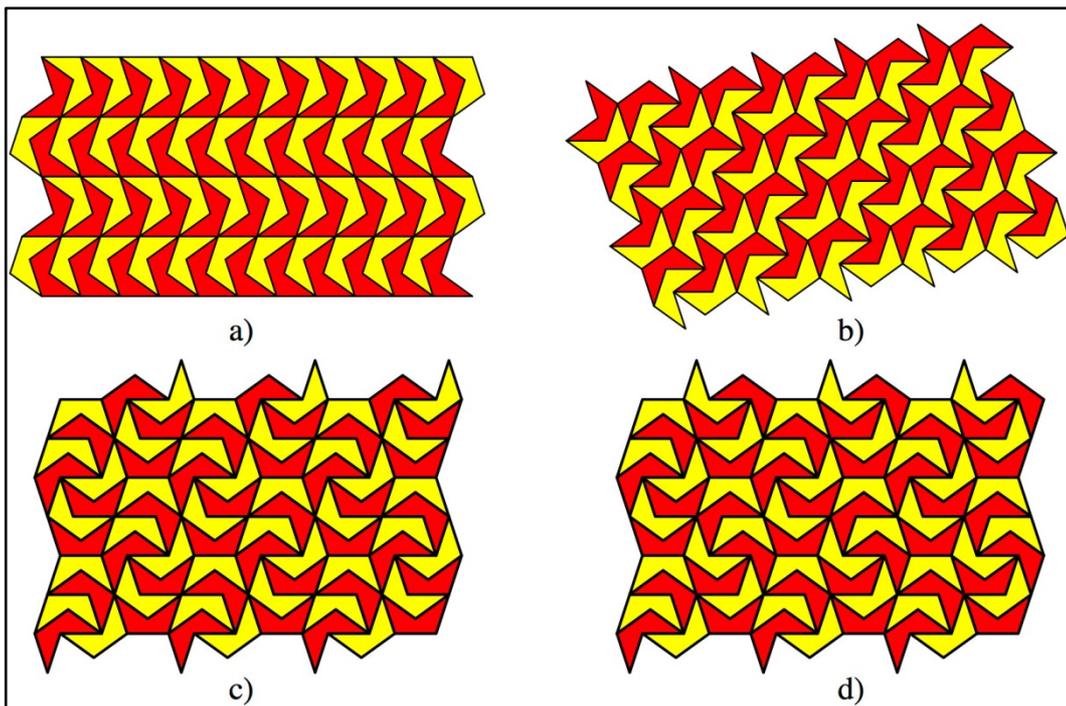


Fig. 6: Tilings

Rings and spirals

The figure 7a gives a constellation with concentric rings. The figure 7b is basically the same, but the colors are exchanged such that there are always two different colors at an edge.

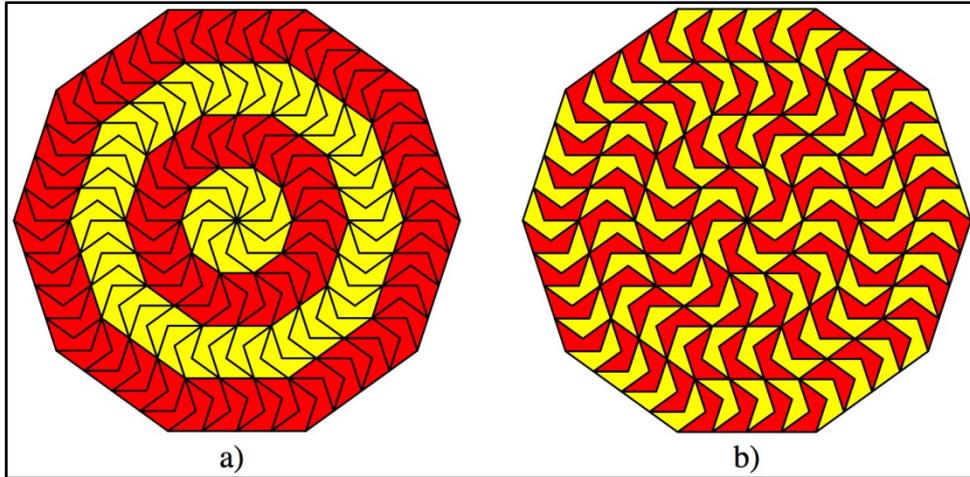


Fig. 7: Concentric rings

First we see nothing special in the figure 8a. But a closer look reveals a spiral (Fig. 8b). The width of the spiral is constant, so we have an Archimedean spiral.

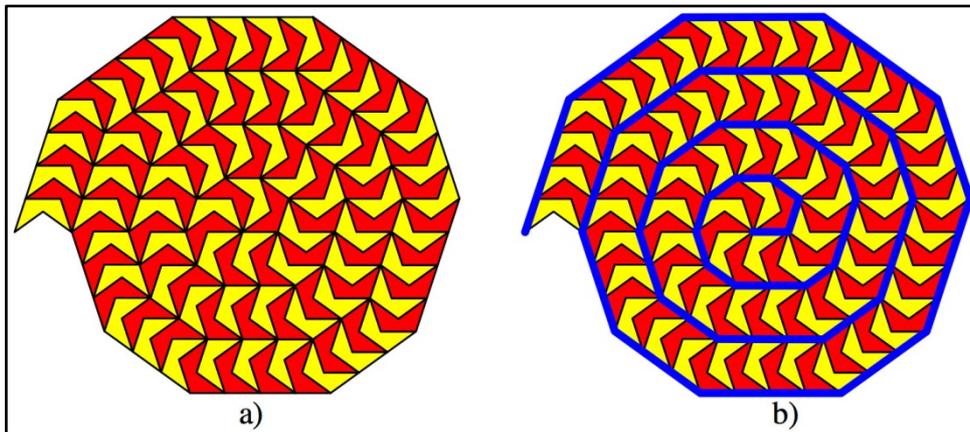


Fig. 8: Spirals

The figure 9 depicts two and ten Archimedean spirals respectively.

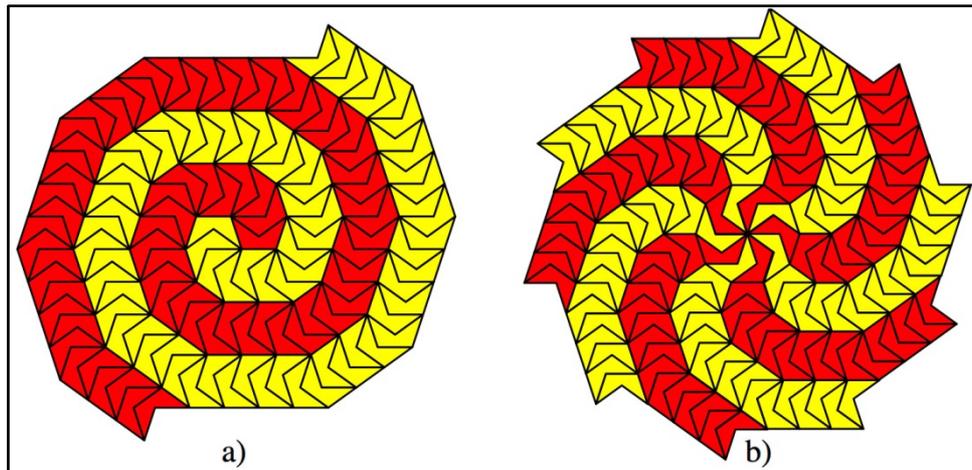


Fig. 9: More spirals

In the three-dimensional space

With twelve regular pentagons as faces we can build the regular dodecahedron (Fig. 10a). Together with the regular tetrahedron, the cube, the octahedron, and the icosahedron it belongs to the five regular Platonic solids.

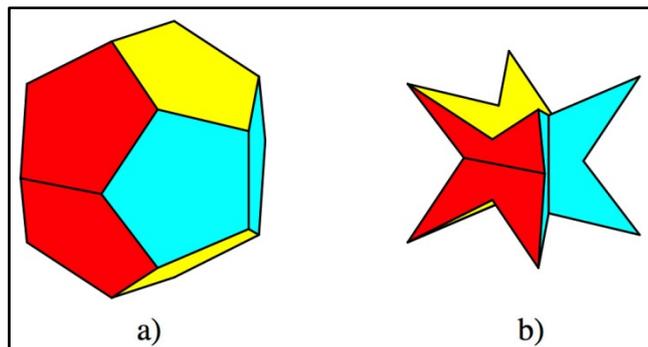


Fig. 10: Regular and semi-regular dodecahedron

With twelve semi-regular pentagons we can build the semi-regular dodecahedron (Fig. 10b).

Relation to the cube

The semi-regular dodecahedron fits obviously in a cube (Fig. 11b). But we can also draw a cube on the faces of the regular dodecahedron (Fig. 11a).

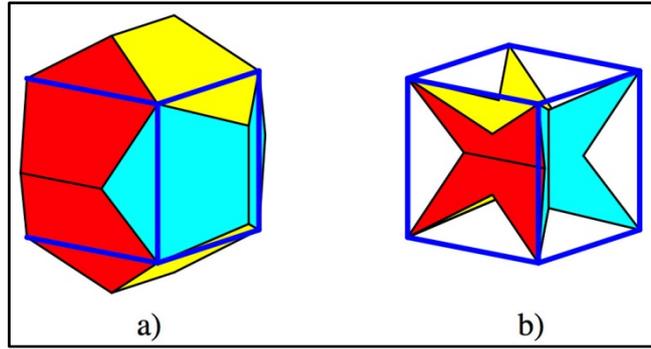


Fig. 11: Relation to the cube

The regular dodecahedron may be seen as a compound of a cube with a hip roof on every face. And surprisingly the semi-regular dodecahedron is just the leftover when we cut away a hip roof at every side of the cube. So we have related construction principles. The regular dodecahedron is positive, the semi-regular dodecahedron negative. The hip roof excess of the regular dodecahedron corresponds to the hip roof deficit of the semi-regular dodecahedron.

The figure 12 depicts the situation between the two dodecahedra.

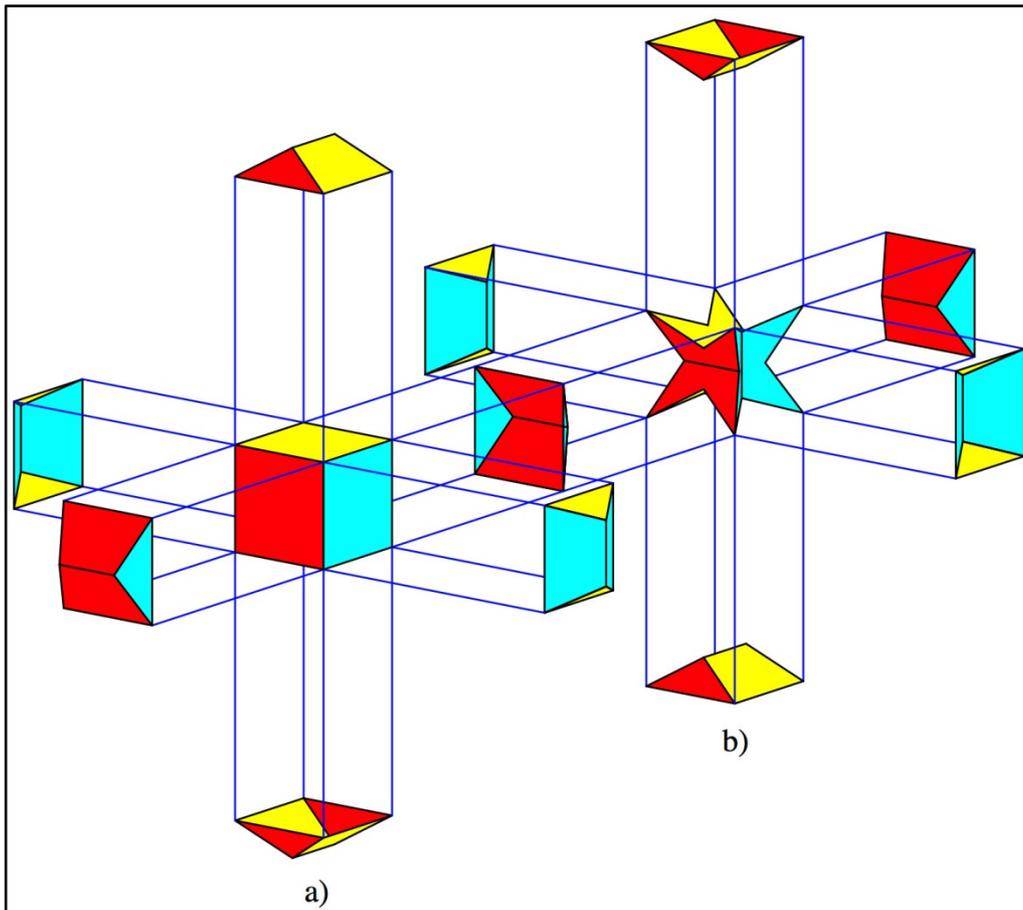


Fig. 12: Hip roofs

The hip roof cut away from the cube at the right (Fig. 12b) goes to the cube at the left (Fig. 12a). We have a situation similar to the situation of anions and cations in chemistry.

A singular hip roof has the dimensions indicated in the figure 13.

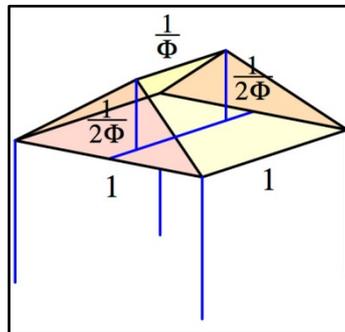


Fig. 13: Dimensions of the hip roof

The notation Φ means the Golden Section (Walser 2001 and 2013) $\Phi = \frac{1+\sqrt{5}}{2} \approx 1.618$. The height of the roof is $\frac{1}{2\Phi} \approx 0.309$.

Eight-pointed stars

The semi-regular dodecahedron is an eight-pointed star (Fig. 14b). But it is different from the similar looking *Kepler star* or *stella octangula* (Fig. 14a).

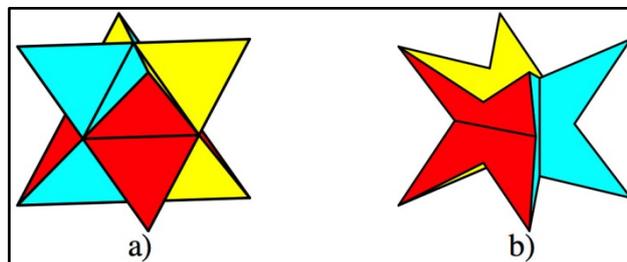


Fig. 14: Kepler star and semi-regular dodecahedron

Symmetry groups and topology

The symmetry groups of the regular and the semi-regular dodecahedra are different. The semi-regular dodecahedron has just the symmetry group S_4 of the regular tetrahedron.

The regular and the semi-regular dodecahedra have the same topology (Fig. 15 and 16). Both have 20 vertices, but in the semi-regular case 12 of the vertices are hyperbolic.

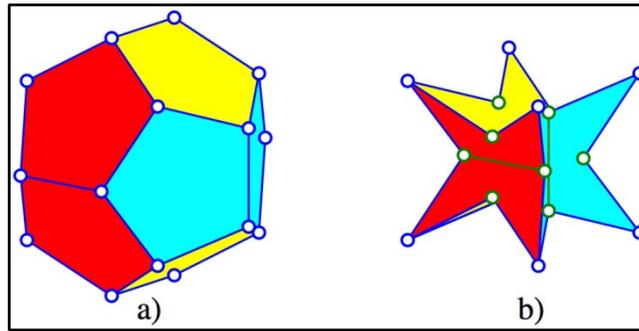


Fig. 15: Vertices, edges, and faces

Both have 30 edges, but in the semi-regular case six of the vertices are like the bottom of a valley. The regular dodecahedron has 12 regular pentagons as faces, in the case of the semi-regular dodecahedron the 12 faces are semi-regular pentagons.

In the diagrams of the figure 16 we see how the vertices and edges are connected. There is no difference between the two dodecahedra.

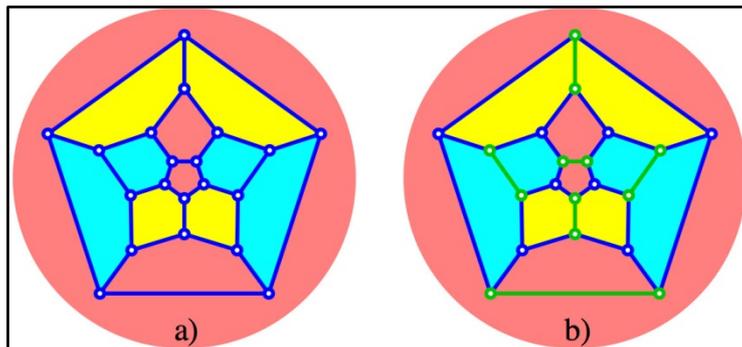


Fig. 16: Diagrams for regular and semi-regular dodecahedron

Model

We can build a paper model of the semi-regular dodecahedron (Fig. 17a).

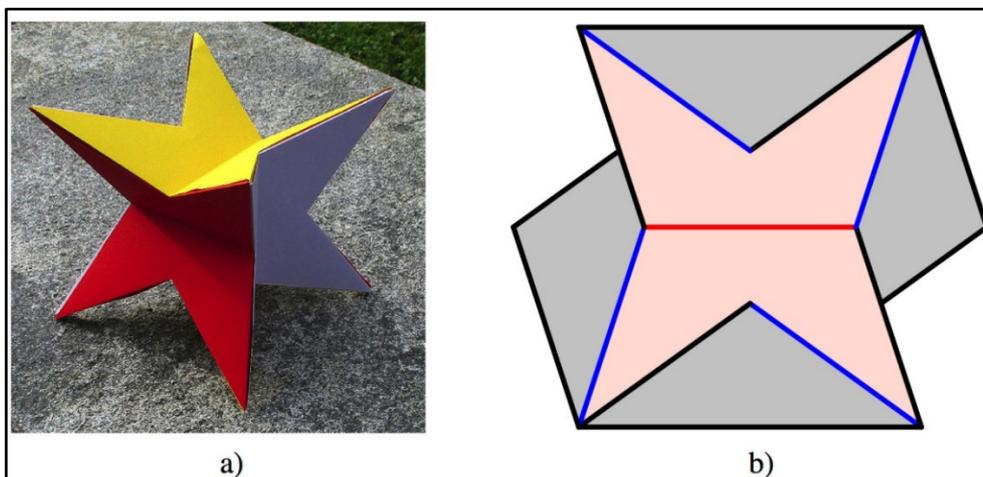


Fig. 17: Paper model

We need 6 parts according to figure 17b. We have to cut along the black lines, fold in the sense of a valley fold along the red line and in the sense of a mountain fold along the blue lines.

The red area remains visible on the outside, and the gray areas can be tucked in or glued. Hint: The model will not be very stable. Therefore I built first a model with reduced size (98%) and the visible model as second layer on it.

Filling the space

Because of the relative situation between the two dodecahedra (Fig. 12) the regular dodecahedron can sit on the semi-regular dodecahedron like the egg on an eggcup (Fig. 18).

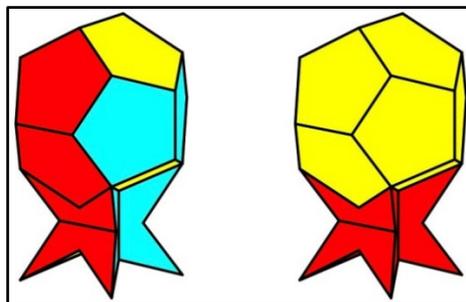


Fig. 18: Eggcup

Neither the regular nor the semi-regular dodecahedra are space fillers (Coxeter 1973, p. 68f). But we can fill the space with regular dodecahedra combined with semi-regular dodecahedra (Fig. 19).

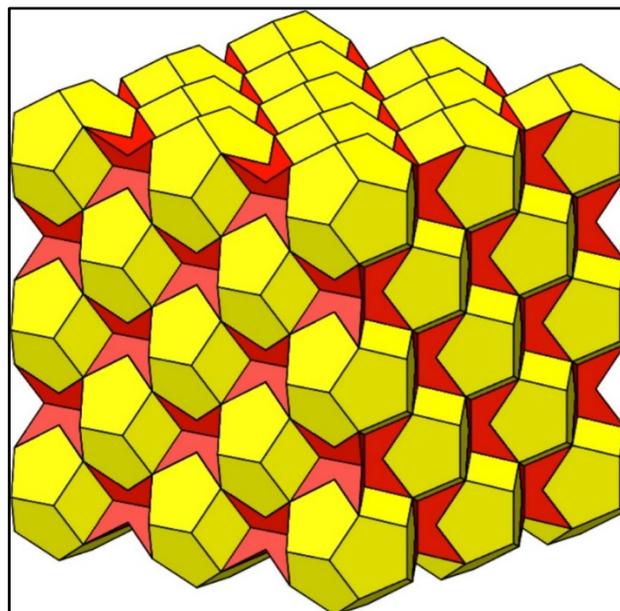


Fig. 19: Filling the space

The figure 19 is somehow the spatial analogue to the figure 3.

The proof of the space-filling property is easy. First, the cube is a space filler. So we fill the space with cubes and color the cubes in black and white like a three-dimensional chessboard. Now we cut away the hip roofs from the black cubes and add them to the adjacent white cubes.

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FROM CENTROID TO EXPLAIN A QUESTION ABOUT THE VOLUME OF CONE

Gao Shuzhu¹ · Chen Weiwei¹ · Zheng Qian¹

Abstract

The volume of cone could be represented by $\frac{1}{3}\pi r^2 h$. A popular question from students and teachers is why the coefficient is $\frac{1}{3}$ rather than $\frac{1}{2}$ like the area of triangle $\frac{1}{2}bh$. This question could be explained from the concept of centroid or the center of mass which is a point of resultant force of gravity of each part of an object. In mathematics we assume a physical object has uniform density, and then its center of mass is the same as the centroid of its shape.

The proposing of problem

In 2-dimensional space, we know that if a right-triangle has the same height as well as the same side with a rectangle, then the area of the right-triangle () is the half of the rectangle. In 3-dimensional space, we also know that a cylinder could be the result of a revolving rectangle; a cone could be the result of a revolving right-triangle. While a popular question among students and teachers come: why the volume of cone should be represented by $\frac{1}{3}\pi r^2 h$ rather than $\frac{1}{2}\pi r^2 h$ like the relationship of area between right-triangle and rectangle. As shown in Fig.1:

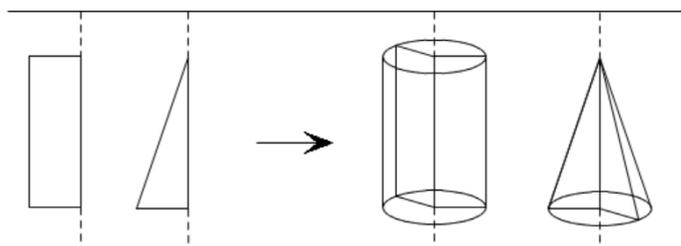


Fig 1 The relationship between solid figure and plane figure

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Another version of this problem is as follows: when we divide a rectangular into two congruent right-triangles 'ABD' and 'BCD'(as shown in Fig.2),then we made the rectangle 'ABCD' revolves round the side 'BC', we cannot get two congruent cones from triangles 'ABD' and 'BCD'.

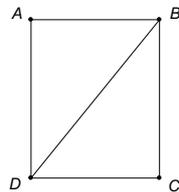


Fig 2 two congruent right-triangles divided by rectangle

Deeper thinking with the factors of the volume

There's no doubt that different figures resulted in different volume, but we also found that it is not the only factor for the volume. Let's suppose a point 'A' revolves around line 'L'(as shown in Fig.3), if the vertical distance between point 'A' and the rotation axis is 3cm (that is to say the radius of the following locus is 3cm), then the circumference of this circle is 6π . However if the distance turns to 4cm, then the circumference of the circle turns to 8π . Thus it can be seen that the distance between the rotator and the rotation axis is another important factor apart from the shape of the rotator.

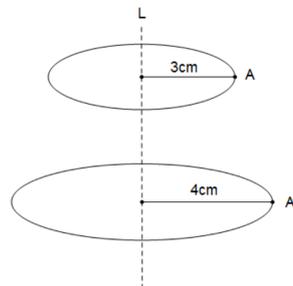


Fig 3 the circumference of different distance between the rotator and the rotation axis

Controlling variables is a good way to allow the clear identification of cause and effect because only one factor is different at the same time, so that the effect of that single factor can be determined. Now we look back to figure 2 and make the figure revolves around the axis 'BC'(5cm) as well as axis 'DC'(3cm) respectively (as shown in Fig.4), then we get two different cylinders, the volumes are 45π and 75π respectively. Thus it shows again and that the distance between rotator and the rotation axis must be another important factor when reference to the volume.

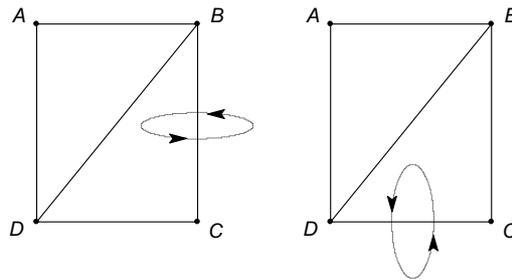


Fig 4 revolving in two ways

The late Greeks Pappus had written in his *Mathematical collections* that :the volume V of a solid of revolution generated by rotating a plane figure F about an external axis is equal to the product of the area A of F and the distance d traveled by its geometric centroid..... Now we call it *Pappus's centroid theorem* (Thomas Heath 1923). This theorem shows two main factor which are the decision for the volume of revolution, "the area of the plane figure A " and "the distance d travelled by its geometric centroid". The centroid we mentioned here is a concept from physics. In mathematics we assume a physical object has uniform density, and then the center of mass is the same as the centroid of the shape. For a centro-symmetric figure, the center of mass is coinciding with the center of symmetry. For example, the centroid of a line segment is the midpoint; for a rectangle, the centroid is the intersection of its diagonals; the center of a circle is also the circles' center of mass. So for all plane figure in mathematics, we could now only focus on its centroid instead, also when we discuss the distance between the plane figure and the external axis, we can use the centroid instead of the plane figure. For the volume of the cylinder in picture 4($AB=3\text{cm}, BC=5\text{cm}$), we now know the distance between the centroid of the rectangle and the axis BC is 1.5cm , according to *Pappus's centroid theorem*, the distance d travelled by its geometric centroid should be " $2 \times 1.5 \times \pi = 3\pi$ ", the area of the rectangle is 15 , so the volume is 45π if the axis is line ' BC ', another one should be $(2 \times 2.5 \times \pi) \times (3 \times 5) = 75\pi$. Then we talk about the situation of triangle.

Centroid of triangle

As we see the cone comes from the revolving triangle revolves around its side, we need to know the location of the centroid of the triangle. Here we need two central lines of triangle ' ABC ', line BE and line AD (as shown in Fig.5). For two triangles, if both the bottom lines and the heights are equal, then their area are equal. So, triangle ABC was divided into two equal-area-triangles by line AD , because of the symmetry of centroid, it must be on line AD , also on line BE in the same way, thus point ' O ' is what we are looking for.

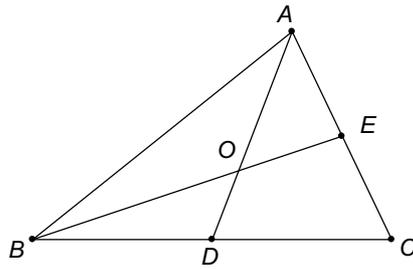


Fig 5 centroid of triangle

For measurement purposes, it is necessary to know where it exactly is. Triangle ADC and triangle BCE are both the half of triangle ABC, thus the area of triangle ADC equals to the area of triangle BCE, get rid of their common part 'quadrilateral OECD', then the last parts triangle AOE and triangle BOD have the same area. In the same way, triangle has the same area as quadrilateral OECD. Join point O to point C (as shown in Fig.6), so we find triangle OBD has the same area as triangle OCD, the area of AOE equals to the area of triangle COE. Above all, the area of triangle AOB is two times of triangle AOE as well as triangle BOD, so the length of line segment BO is twice as line segment OE, the length of line segment AO is twice as line segment OD. We included the position of the centroid of any triangle as follows:

- a. For any of the triangle, the centroid of the triangle is the intersection of the three medians of the triangle (each median connecting a vertex with the midpoint of the opposite side).
- b. The centroid divides each of the medians in the ratio 2:1, which is to say it is located $\frac{1}{3}$ of the distance from each side to the opposite vertex (as shown in Fig.7).

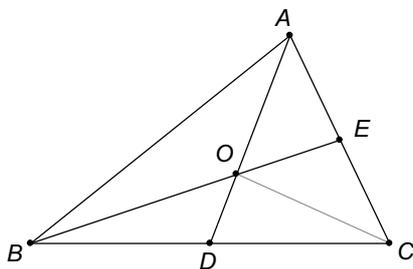


Fig 6 the certification of triangle centroid

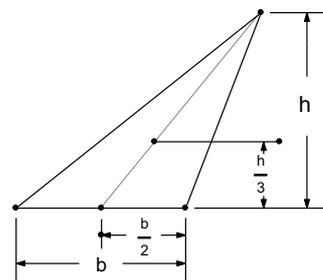


Fig 7 nature of centroid

The answer to the original question

Look back to the former question: we made the rectangle 'ABCD' revolves round the side 'BC', why we cannot get two congruent cones from the two congruent triangles 'ABD' and 'BCD'?

In order to show you the answer to the original question well, we need to add three auxiliary lines: the diagonal line 'AC'; two central lines of triangle 'ABD' (line DF) and triangle 'BCD' (line BE) (as shown in Fig.8). In this way, we find point M as the centroid of triangle ABD and point N as the centroid of triangle BCD. As the distance between the centroid N and the axis BC, the length of line segment NH is obviously shorter than the length of MG. According to *Pappus's centroid theorem*, it is no wonder to see two different cones comes from even two congruent triangles.

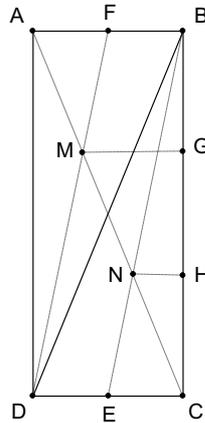


Fig 8 analysis of the problem

And then, according to "The corresponding edges of similar triangles are proportional", we know the length of line segment $NH = \frac{1}{3} AB$, the length of line segment $MG = \frac{2}{3} AB$, so the length of $NH = \frac{1}{2} MG$. For any circumference of a circle is proportional to its radius, so the distance d_1 travelled by the geometric centroid M is two times of the distance d_2 travelled by the geometric centroid N. At last, we say, when we have two congruent triangles, the relationships of the radius of gyration is double, and then the relationship of the volume of the two cones are double according to *Pappus's centroid theorem*. For any triangle that has the same height as well as the same side with a rectangle, the volume of the cylinder comes from the revolving rectangle must be three times as a cone from the triangle.

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COMPETENCIES, CURRICULA AND INTERDISCIPLINARITY: AN ANALYSIS OF A CURRICULUM DEVELOPMENT PROCESS IN MATHEMATICS AND SCIENCE EDUCATION

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Abstract

Teachers in science and mathematics education face many challenges as curricula around the world turn towards STEM education as a way of ensuring that students acquire competencies that transcend traditional subjects. However, implementing competency-oriented teaching has proved challenging. KOMPIS was a project that aimed at developing ways of dealing with some of these challenges. In this paper, we present a two-dimensional model derived from KOMPIS that has been demonstratively useful in supporting competence-based curriculum development and teacher planning.

Introduction

KOMPIS was a combined research and development project conducted in the years 2009-2012. The project was based on a productive collaboration between 16 teachers from four schools, three university researchers and three teacher educators, split into three subject groups – math, science and Danish. The experimental teaching, which constituted the focal point of the project, focused on conducting competency-oriented teaching in lower secondary classrooms (grades 7-9). The results of the collaboration included the development and testing of models, concepts and teaching practices that evolved from the ongoing experiments. Some of these models and concepts proved to be incorporated in significant ways in the current national Common Objectives in Danish public schools, which was developed and implemented in the years 2013-2015.

The specific work and results of KOMPIS have been described in detail in previous work (Højgaard et al., 2010 and Sølberg et al., 2015). In this paper, we focus on the curricular aspects of this research and development process within the subjects of mathematics and science. Firstly, the framing of KOMPIS is

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described along with an analysis of the concepts of competence and competencies as curricular building blocks. Secondly, we propose a two-dimensional framework for a binding incorporation of competencies in a curriculum, exemplified by the actual curriculum frameworks tested out in KOMPIS. Finally, we discuss the potentials of such curriculum frameworks for the pursuing of interdisciplinary educational ambitions.

Challenges motivating KOMPIS

Teaching STEM

According to the Future of Jobs report from the World Economic Forum, jobs of the future will become increasingly complex (World Economic Forum, 2016). Demands for future citizens as well as the workforce of the future depends on education of students that can deal with complex problems that transcend traditional disciplines. Dealing with complex problems through interdisciplinary approaches, mirroring real world problems, requires innovative ways of planning and organizing teaching (Cai, 2011, Hall 1995).

A STEM approach could potentially be a useful tool in meeting future demands and as a vehicle for curricular development. STEM is typically understood as the integration of Science, Technology, Engineering and Mathematics in education, but it can be argued that STEM covers a broader spectrum of knowledge areas ranging from the segregated domain-specific areas such as science and mathematics to more integrated domain-general topics such as health care or climate change (Nadelson & Seifert, 2017). In spite being a dominant trend in educational policies, understanding the possible gains and potential pitfalls of implementing STEM remains a challenge. However, small scale attempts to create cross-disciplinary teaching in schools that help students learn how to deal with complex problems are not new. Competency-oriented approaches have become more prominent in educational policies and if implemented properly, these approaches hold a lot of promise for future curriculum development. An example of such policy is the Next Generation Science Standards from USA (National Research Council, 2011), where the authors propose to organize teaching around Disciplinary Core Ideas as well as cross-disciplinary Practices and Concepts.

While there are many potential benefits to adopting a STEM approach throughout our educational systems, such strategies also pose many challenges for implementation such as providing necessary teacher education and in-service training, re-organizing schools to accommodate new forms of teaching, developing updated assessment schemes and many other challenges (National Research Council, 2014). However, in this article, we will focus on the challenges

for science and mathematics teachers in planning teaching that can not only accommodate ambitions for future citizens but also address curricular requirements.

A recent review of STEM education research (Nielsen, 2016) points to many curricular challenges for STEM teachers such as including inquiry-based teaching; preparing student to deal with socio-scientific issues; doing practical and experimental work; providing opportunities for learning in informal setting, incorporating engineering, design and technology in their teaching; using IT etc. While these are all in and of themselves worthy goals, they pose a massive challenge for most teachers to deal with in their teaching. Add to that the challenge of students having to navigating the massive amounts of subject matter within STEM teaching is likely to be struck with what has become known as *syllabusitis*.

Syllabusitis

What constitutes a subject, e.g. mathematics or biology? Many things, of course, but we feel convinced that everyone will agree, that – among other things – mathematics relates to certain objects, concepts and procedures that we (tautologically) consider as mathematical, and similarly for biology. Many people use this relation to subject matter to characterize the subject (cf. Blomhøj & Jensen, 2007, p. 46-47, which this and the next section is based on). “Mathematics is the subject dealing with numbers, geometry, functions, calculations, etc.” and “biology is the subject dealing with animals, plants, evolution, ecology, etc.” is not a rare type of answer to the questions of what constitutes mathematics and biology respectively.

What, then, does it mean to master a subject? With reference to the above, it is tempting to identify mastering mathematics/biology with proficiency in mathematical/biological subject matter. However, this belief, if transformed into educational practice, is severely debilitating for students’ ability to make reason of the subjects and to apply them in future contexts. The debilitating effect is potentially severe enough that the phenomena has been given a name that evokes images of a disease, namely *syllabusitis* (Lewis, 1972; Jensen, 1995). *Syllabusitis* represents the condition that occurs, when curriculum goals, assessment schemes, textbooks etc. are prescribed in a way that fails to acknowledge important aspects of what it means to master a subject, e.g. mathematical reasoning or conducting biological experiments to mention a few. Most curricula include ambitions that the aims of a given subject is to make students *better* at the given subject. However, a curriculum infected by *syllabusitis* tends to focus only on the ability to reproduce subject content and therefore fails to set an appropriate level of ambition and puts the teacher in a position where they

struggle to cover the prescribed content. Hence, it becomes an important challenge to address the issue of describing what mastering a given subject means in a way that can prevent syllabusitis.

The KOM project, competence and mathematical competencies

The issue of syllabusitis was one of the main issues to be addressed when Mogens Niss proposed to conceptually define mathematical competencies as a tool for developing mathematics education (Niss, 1999). The so-called KOM project (Niss & Jensen, 2002), running from 2000-2002, thoroughly introduced, developed and exemplified mathematical competencies at all levels of education from primary school to university (cf. Niss & Jensen (to appear) for an actual presentation and analysis of the project and an English translation of the original report).

The definition of the term “competence” in the KOM project (Niss & Jensen, 2002, p. 43) was semantically identical to the one we use: *Competence* is someone’s insightful readiness to act in response to the challenges of a given situation (cf. Blomhøj & Jensen, 2003). The important move is, however, to focus on *a mathematical competency* defined as someone’s insightful readiness to act in response to *a certain kind of mathematical challenge* of a given situation, and then identify, explicitly formulate and exemplify a set of mathematical competencies as independent dimensions spanning full range of mathematical competence. The core of the KOM project was to carry out such an analysis, of which the result is visualized in condensed form in figure 1.

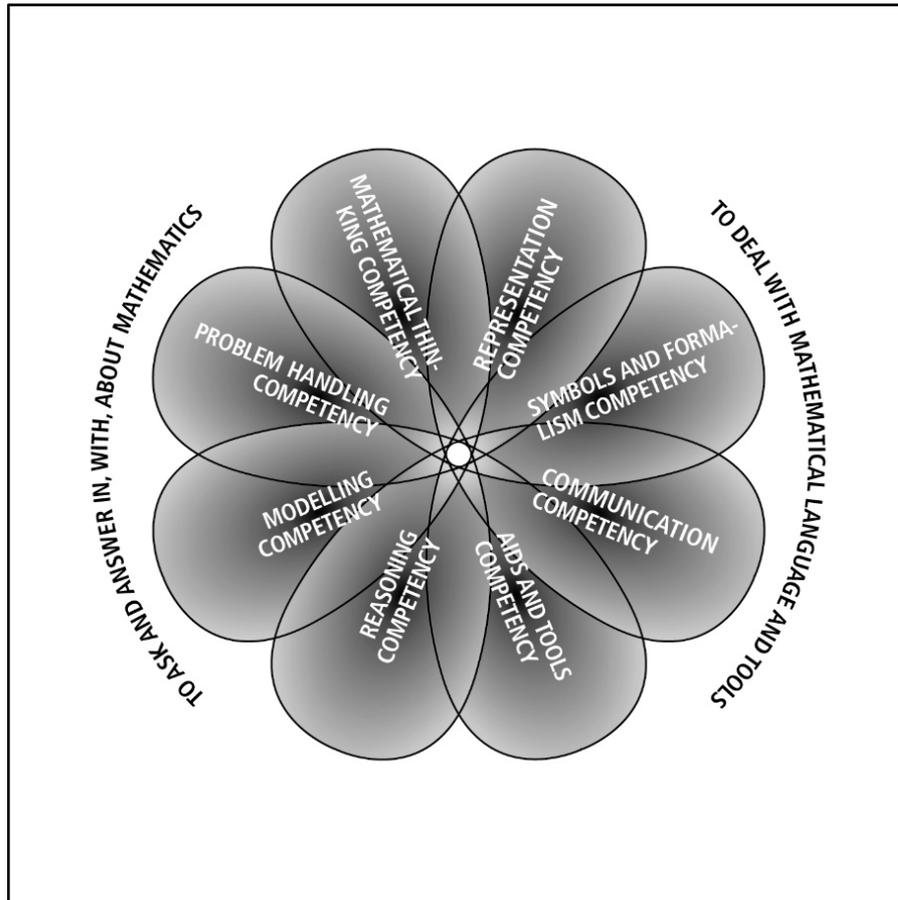


Figure 1: A visual representation – the “KOM flower” – of the eight mathematical competencies presented and exemplified in the KOM report (Niss & Højgaard, to appear).

Such a set of mathematical competencies has the potential of replacing the syllabus as the focus of attention when working with the development of mathematics education, simply because it offers a vocabulary for a focused discussion of what it means to master mathematics (Jensen, 2007). Often when a syllabus attracts all the attention in a developmental process, it is because the traditional specificity of the syllabus makes us feel comfortable in the discussion.

A framing content description for mathematics

Following the approach of the KOM project, the endeavour to incorporate subject specific descriptions of competencies in mathematics curricula should primarily be guided by an attempt to fight syllabusitis. In this perspective, it was important to focus on the interplay between subject specific competencies and the subject specific matter traditionally described in the syllabus.

In the KOM report the proposal for such an interplay is to separate subject specific competencies and subject matter areas as two different dimensions of content (Niss & Højgaard, to appear), and subsequent research and development

work prior to the KOMPIS project supported the importance of such an approach to curriculum development (Jensen, 2007).

In the KOMPIS project we continued the development of this approach, based on the general hypothesis that a goal oriented (yearly) planning is supported by a curriculum that is systematically developed for enhancement of transparency, offering itself as a thinking tool for the teachers. The model in figure 2 was a proposal for such a transparent representation of the core of the 2009 National Standards for mathematics in Denmark (Ministry of Education, 2009). The two dimensions *competencies* and *subject matter areas* span the content, with a traditional emphasis on applications of mathematics represented by emphasizing mathematical modelling competency as being particularly important.

Subject matter area Competency	Numbers and algebra	Geometry	Statistics and probability
Mathematical thinking competency	[Empty cell]		
Mathematical problem handling competency			
Mathematical modelling competency			
Mathematical reasoning competency			
Mathematical representation competency			
Mathematical symbols and formalism competency			
Mathematical communication competency			
Mathematical aids and tools competency			

Figure 2: A two-dimensional model for description of the content of grade k-9 mathematics teaching (Højgaard et al., 2010, p. 15)

In a curricular perspective, the defining point of such two-dimensional content models is that the subject specific competencies can function as “the missing link” between the overarching purpose of an education and a concrete syllabus, by pointing out what types of challenges the students must be able to *act* in relation to.

A framing content description for science

Prior to the KOMPIS project there had not been a development within science towards a common conceptualisation of competencies akin to the more comprehensive process that the mathematics curriculum had undergone. The closest approximation to a description of science competences could be found in the so-called FNU report (Andersen et al., 2003, p. 41), which described four overarching competencies (author’s translation):

- *Science-based investigative competency* (experimenting, data collection, using scientific methods etc.)
- *Science-based modelling competency* (building and interpreting models, making predictions, reducing complexity etc.)
- *Science-based representation competency* (using and reading symbols and representations, differentiating between different scales and levels of abstraction etc.)
- *Science-based perspective competency* (relating to historical development of science and society, critical reflection on use of technology, nature of science etc.)

These descriptions comprised the conceptual starting point for the science group going into KOMPIS.

Based on the four competencies in the FNU report, the science group tried to build a model corresponding to the one for mathematics in figure 2. However, the science in the mandatory part of the Danish school system consists of four different science subjects with different teacher groups and traditions. This division of science into different subjects was an added challenge to the model in that it had to be relevant for all the science teachers at the same time to function as an effective planning tool for the group.

This led to another challenge, which was that the concept areas in the model could not be readily defined in the same way as they were in mathematics. At face value, the model had to cover the content of all four subjects at the same time

Planning competency-oriented teaching

Planning of teaching can be considered as the process of making decisions regarding the content (Larsen, 1969). In other words, planning deals with the question: What are the students to learn? In this sense, planning of teaching inevitably boils down to one dominating dimension: Time. Since time is a scarce resource when teaching takes place within a formal educational system like compulsory schooling, a more interesting version of the question mentioned above is: *When* are the students to learn what?

The two-dimensional content models in figure 2 and 3 do not immediately assist with the challenge of developing yearly teaching plans. More didactical considerations are necessary when planning from a two-dimensional content structure to a one-dimensional plan for the year, and the emphasis on competency development in the model often requires more prolonged and successive teaching time making it necessary to plan teaching in modules wherein the students get to focus on few goals.

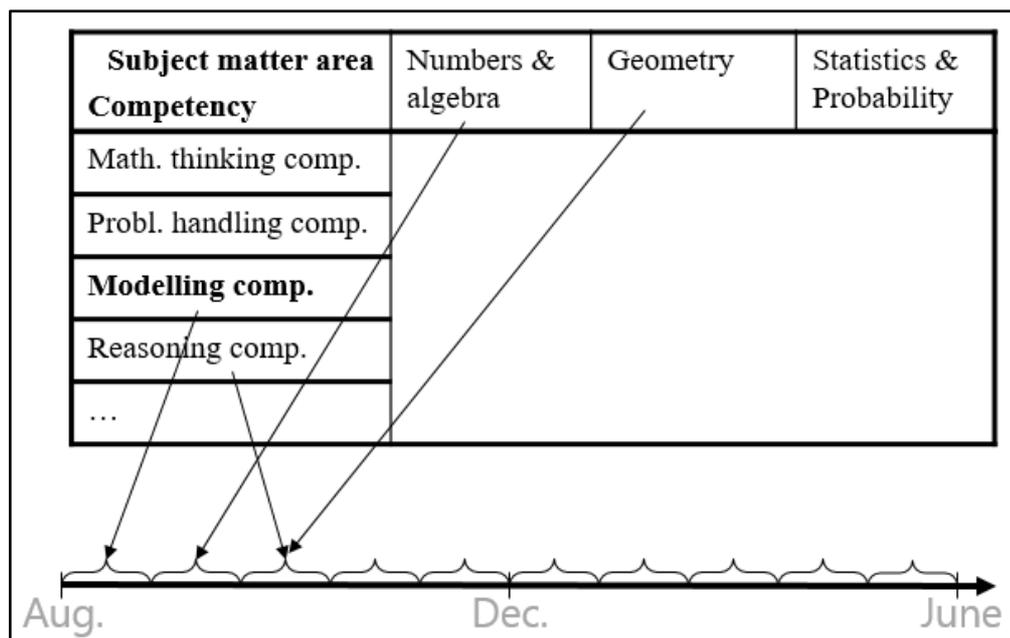


Figure 4. The two-dimensional model used as a tool for planning of teaching by creating modules of some weeks in duration, each appointed with explicitly stated learning objectives consisting of subject specific competencies and/or objectives related to a subject specific concept.

This approach can be modelled by dividing the year up into more-or-less distinct modules and deciding which areas of subject matter, competencies or combination of the two to focus on as illustrated in figure 4. This model functions as a planning and reflection tool for teachers as well as a model that teachers can use when they discuss what the objectives of a given teaching session is with

colleagues and students. As illustrated in the model, an important part of implementing competency goals in teaching is to plan for extended periods of time, where students and teachers can focus on specific competency goals (Højgaard, 2010).

An interdisciplinary perspective

Similar to the way you can plan mathematics or science teaching according to the two-dimensional model, you can also combine them to deal with some of the challenges for teachers using a STEM approach. By expanding the two-dimensional model to combine science and mathematics (and potentially any other subject as well), you can plan the year in modules that draw from the areas of subject knowledge and competencies of both fields.

Combining two-dimensional models from more than one subject potentially threatens some of the advantages of the two-dimensional approach by expanding the list of potential subject matter and competencies to focus on. However, in some cases, the competencies or subject matter overlaps in a way to can give rise to interdisciplinary modules as illustrated below.

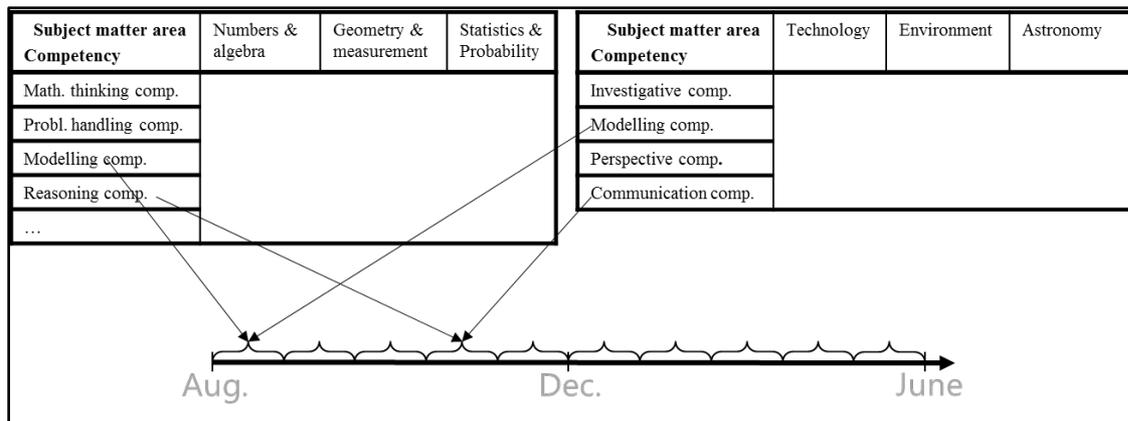


Figure 5: A model of an interdisciplinary approach to the planning of competency-oriented mathematics and science teaching.

Using the two-dimensional models from each subject, teachers can readily examine differences and overlaps within the subject and use such an analysis to identify constructive overlaps such as modelling in the example above. Working with modelling in mathematics is not necessarily the same as working with modelling in science, but by setting aside time in the yearly planning to allow students to work on modelling across subjects could add depth and perspective to their understanding of modelling. In this way, the potential benefits of working with modelling in an interdisciplinary approach should become visible to both teachers and students, and we can begin to foster student competencies

more systematically. This, we believe, is a crucial step if we are ever to implement STEM in schools.

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ADDRESSING MATHEMATIZATION OBSTACLES WITH UNFORMALIZED PROBLEMS IN PHYSICS EDUCATION

Martin Niss¹

Abstract

Solving a physics problem requires that the problem solver either implicitly or explicitly structure the problem situation in such a way that she can set up the mathematical equations based on the relevant physics. This part of the mathematization process has been shown to cause obstacles for students (Niss, 2016). In the paper, we show how the students' ability to perform this mathematization process can be trained by using so-called unformalized physics problems. Some examples of how this training can be done are provided from a course on problem solving in physics taught at Roskilde University.

Introduction

While Thomas Kuhn was mainly preoccupied with philosophy and history of science, he also had some pertinent ideas about physics education. Kuhn (1970) pointed out that it is in fact a quite complex task to apply equations like Newton's second law $\mathbf{F} = \mathbf{ma}$ to particular physical situations, i.e. to identify the relevant force, mass and acceleration in such a way that the equation can be used. In fact, the equation takes on different forms depending on the situation:

- For the case of free fall, $\mathbf{F} = \mathbf{ma}$ becomes $m\mathbf{g} = m \frac{d^2s}{dt^2}$.
- For the simple pendulum, $\mathbf{F} = \mathbf{ma}$ is transformed into $mg \sin \theta = -ml \frac{d^2\theta}{dt^2}$.
- For a pair of interacting harmonic oscillators the equation becomes two equations, the first of which may be written

$$m_1 \frac{d^2s_1}{dt^2} + k_1s_1 = k_2(s_2 - s_1 + d)$$

So, even though most theories of physics are formulated in mathematics, such as $\mathbf{F} = \mathbf{ma}$, the process of setting up a mathematical description of a physical situation based on physics is complex and difficult. Kuhn's point has been

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corroborated by several empirical studies in physics education (see Niss, 2016) and hence the evidence indicates that mathematization in physics is something that needs to be learned.

Kuhn was perhaps even more ground-breaking in his answer to the question: How does the student learn to apply equations like $F = ma$ to new situations? His answer: “The student discovers, with or without the assistance of his instructor, a way to see his problem as like a problem he has already encountered.” (Kuhn, 1970, p. 189). So, according to Kuhn an important aspect of the mathematization competency is about seeing problems as like each other. The process of learning to see problems as like each other is often tacit in that it is not acquired by exclusively verbal means but upon seeing examples of how the problem-solving process plays out in the specific setting.

The mathematization process can be more or less involved and depends on the particular formulation of the problem in question. While standard problems in physics (“back-of-the-chapter” problems in textbooks) train the students to apply the equations of physics equations to simple situations, these problems are typically formulated so that the mathematization aspect is more or less explicitly given. Hence, they rarely train the students’ ability to set up equations in more complex situations and to see problems with different physics content but with the same mathematization strategies as like each other. In this paper, I’ll discuss why and how we use unformalized physics problem in a physics course at Roskilde University to teach students how to mathematize situations. The course was conceived by Jens Højgaard Jensen, who also developed the special kinds of problems used in the course. The course has been taught since 1976; at present, there is a bachelor level and a masters’ level version of the course.

Mathematization

The modeling cycles of mathematics education, such as the one in figure 1, that are used to characterize the process that a person goes through when modeling typically include a mathematization step, where the elements of the extra-mathematical system and the question(s) sought to be answered are given mathematical formulations. This description of mathematization process in modeling can be used to characterize the quite similar mathematization process in problem solving. Hence, the problem-solving process can be seen to consist of the following steps (Niss, 2016):

1. An extra-mathematical “systematization”, leading to the system, i.e., the objects in the real-world domain and the relations between them.

2. A mathematization, leading to a translation of the components and questions of the non-mathematical system into a mathematical universe.
3. A mathematical problem-solving process within the mathematical universe.
4. An interpretation of the mathematical results in terms of the original situation

A solid finding of mathematics education research is that “while knowledge of and skills in ‘pure’ mathematics are, of course, necessary for an individual’s ability to deal with models and to perform modelling, such knowledge and skills are far from sufficient for that undertaking.” (Niss, 2012, p. 51) In particular, the mathematization part of modeling is known to cause difficulties and needs to be trained in mathematics education if seen as an educational goal in itself. Similarly, we should expect that mathematization in physics problem solving is something that needs to be trained *per se* in view of Kuhn’s point of view above. However, traditional physics problems are formalized, so that their solution rarely requires mathematization. Consequently, they do not train this skill.

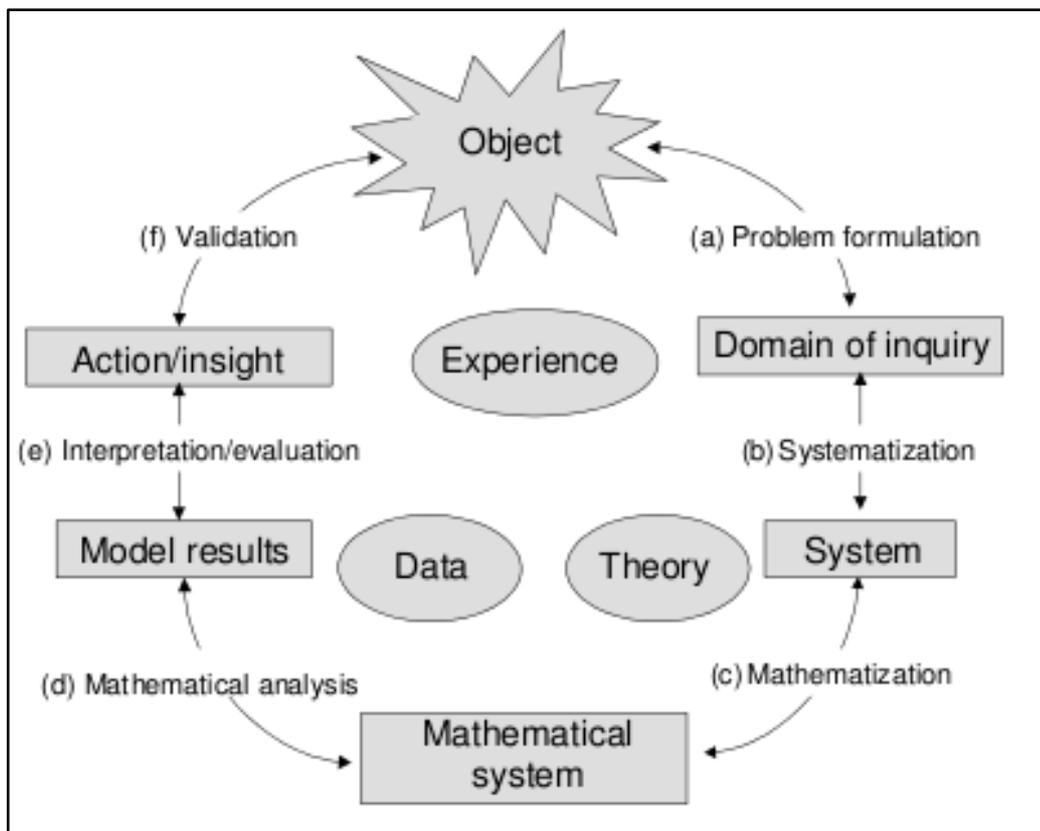


Figure 1: An example of the modeling cycle in mathematics education (Blomhøj, 2004)

Unformalized problems

Unformalized problems are problems where a major challenge is to formalize the problem, that is reformulating the problem in mathematical and physics terms; figure 2 shows the process. An example of unformalized problem is the wind turbine problem: “Give an estimate of the power delivered by a wind turbine. Justify your answer.” Here the problem solver has to reformulate the problem into something like the problem in figure 3.

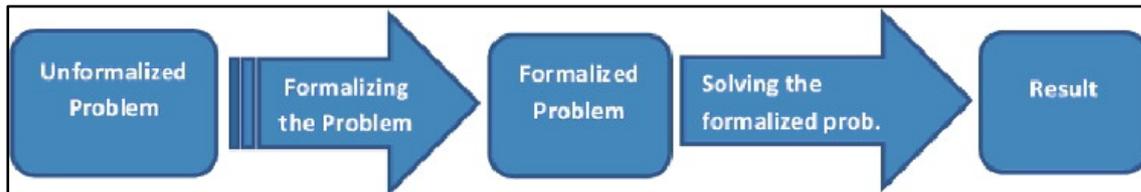


Figure 2: The process of solving an unformalized problem (Jensen, Niss and Jankvist, 2017)

A wind turbine converts kinetic energy of the wind to, e.g., electrical energy. The wind velocity is denoted by v and the density of air by ρ .

1. What is the kinetic energy of the wind per unit volume?

The area of the turbine's blades overflowed by wind is called A . The blades are perpendicular to the wind direction.

2. How large a volume passes this area in time Δt ?
3. How much kinetic energy reaches this area in time Δt ?

The turbine cannot deliver more power (energy per unit time) than the kinetic energy contained in the amount of air that reaches the area overflowed by air per unit time (corresponding to a total halt of the air at the turbine).

4. What is the equation for the upper limit for the wind turbine expressed in terms of A , v , and ρ ?

Figure 3: Elaborate version of the wind turbine problem

Mathematization of unformalized problems can be difficult; Niss (2016) has documented that mathematization can be a major obstacle when solving unformalized physics problems. He identified four obstacles:

1. lack of identification of a mathematical object;
2. an insufficient systematization for mathematization;

3. identification of a mathematical object that is difficult to use for solving the problem;
4. a systematization that makes the mathematical analysis too complicated.

He argued that these obstacles are related to the unformalized character of the problems involved. This means that unformalized problems can challenge the problem solver when it comes to the mathematization and hence such problems can potentially be used for the training of mathematization competency.

Mathematization and the wind turbine problem

The mathematization challenge involved in unformalized problems can be illustrate for the wind turbine problem. The physical analysis of what is going on from a physical point of view is that the energy harvested by the turbine is due to loss of kinetic energy of the wind. An upper estimate of the power delivered can be given by assuming that all the kinetic energy is converted into electrical energy. The next step is to apply the general equation for kinetic energy $K = \frac{1}{2}mv^2$ to the situation. We will focus on this step, even though the physical analysis can be difficult for some students.

For simplicity, we assume that the direction of the air is at a right angle to the plane of the turbine rotors.

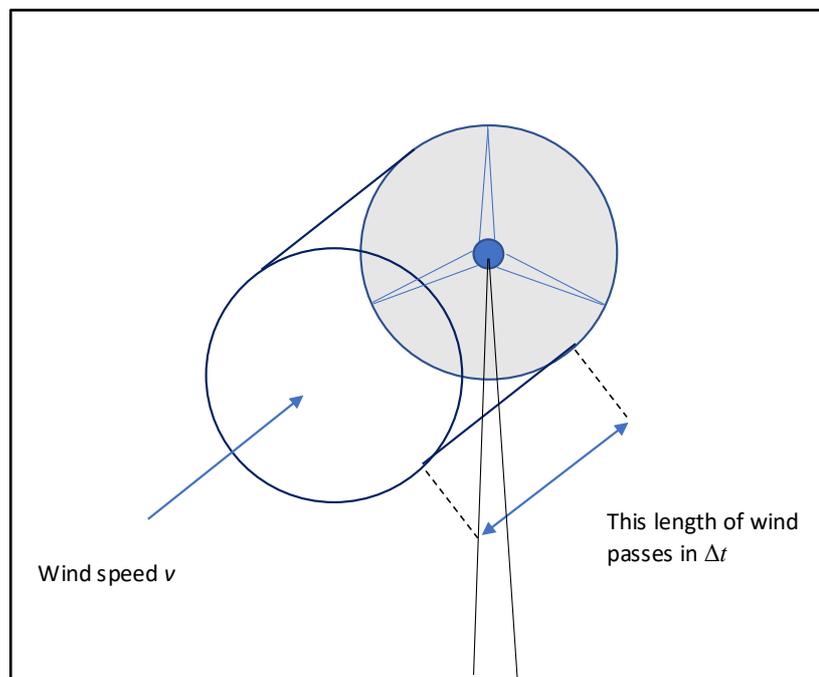


Figure 4: The mathematization of the wind turbine problem.

If the wind speed is v , then a volume of air of $V = v \cdot \Delta t$ will pass through the cross-sectional area A of the rotors during the time interval Δt . This means that the mass

$$m = \rho \cdot V = \rho \cdot A \cdot v \cdot \Delta t$$

passes through the turbine rotors of cross-sectional area A during time Δt . Hence, the power is

$$P = \frac{K}{\Delta t} = \frac{1/2 m \cdot v^2}{\Delta t}$$

By plugging in the mass and doing a mathematical analysis, we find that

$$P = 1/2 \rho \cdot A \cdot v^3$$

When we interpret this result in terms of the original situation, we see that the maximum power that the wind turbine can deliver is proportional to A , ρ and v^3 .

As shown above a major part of the solution process of this problem is to mathematization situation. This is not to neglect that the physical analysis of the situation may cause difficulties as well, but for this problem as well as others it is possible to separate the physical analysis from the mathematization phase.

How mathematization obstacles are addressed in the course

We use unformalized problems to train the students' mathematization competency in the course on problem-solving at Roskilde University. What we do more precisely is that we let the students go through the following steps: They first try to solve an unformalized problem (e.g., the wind turbine problem) requiring one type of mathematization. If they get stuck and most do at first, they get an elaborate version of the same problem – see figure 3 for an example of an elaborate version of the wind turbine problem. The point is that the elaborate problem scaffolds their learning process and highlights the importance of the mathematization process. When the students have solved the original problem, perhaps in the elaborate version, they can try the full-blown solution process on other problems that require similar mathematizations such as the following problems:

1. How does the width of the water column from a water tap change as we go down the column?
2. What is the height of water of a 40 W water fountain?

While the physics of these problems is quite different from the wind turbine problem, the problem solver has to mathematize the situation in a similar way, i.e. focus on the flow of an incompressible fluid through a cross-sectional area. This means that this particular type of mathematization strategy is trained by a

scaffolding procedure and the students learn to see problems that are different in terms of the physics as the same in terms of the mathematization.

Other types of mathematizations can be trained in this way, e.g., the use of differentials which is so important in physics (see, e.g., Artigue, Menigaux & Viennot, 1990).

Conclusion

At Roskilde University, we have quite some success with training mathematization competency using unformalized physics problems in physics course on problem solving. We use unformalized problems and elaborate problems in combination to scaffold the students' development. Hence, we help the students to see problems like each other, in Kuhn's sense, with respect to mathematization even in situations where the physics differs.

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LINKING MATHEMATICS AND BIOLOGY EDUCATION BY MATHEMATICAL MODELING – AN IN-SERVICE TEACHER TRAINING PROGRAM

Claus Michelsen¹

Abstract

Sixty eight upper secondary teachers in mathematics and biology respectively, and three scientific staff members from Laboratory of Coherent teaching and Learning participated in the one-year professional development program “MathBio in the study package”. The overall aim of the program was to enable the teachers to implement interdisciplinary teaching sequences in mathematics and biology in their daily classroom practices. It was the core idea of the program to involve teachers in design, implementation, and evaluation of innovative teaching sequences dealing with a wide range of aspects of mathematics and biology. During the program the teachers shared their didactical reflections, and other issues relevant for the development of the interdisciplinary teaching sequences on Padlets. Based on analysis of the Padlets we report about the teachers’ views of the challenges, limitations and potentials of interdisciplinary modeling activities between mathematics and biology. The experience from the program is that the teachers creatively explore and forge interdisciplinary relationships between mathematics and biology, but that curriculum, time, missing support from the school management and knowledge of the other discipline are obstacles to interdisciplinary teaching.

Introduction

Since 2005 the Danish upper secondary education is organized in specialized so-called study packages containing compulsory subjects, core subjects, and elective subjects. An important feature of a package is that the core subjects form a coherent program, which is ensured by a closer interaction between the subjects. Some of the packages include mathematics and biology as core subjects. To fulfill the objective of coherence in the subject packages interdisciplinary teaching across mathematics and biology is demanded. Mathematical textbooks and

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national tests contain an enormous amount of mathematics exercises disguised as biological problems. The exercise below from a Danish national test in mathematics illustrates this:

Exercise 7

In a model, the relationship between a particular type of trout's length x (measured in cm) and its annual increase in length y (measured in cm / year) is described by

$$y = -x + 21,8$$

- a) Determine the yearly increase in length of a 10 cm long trout.
- b) Apply the model to determine x , when $y = 0$, and offer an interpretation of this number.

(Studienet 2012)

To solve this exercise only mathematical skills are needed, and it is questionable if the biological context makes sense to the students struggling with the exercise. Anyhow, the exercise can be solved without activating the students' biological knowledge. To a certain extent the exercise is exemplary for the current standards of connection between mathematics and biology in Danish upper secondary education.

However, this absence of distinct interdisciplinary connections between mathematics and biology is not only a Danish problem. As a rule the connections between mathematics and biology in the classroom are weak (Jungck 1997, Cox et al 2016), and the process of connecting the two subjects should start with the education of teachers (Šorgo 2010). Connecting mathematics and biology is about change, not for the sake of change but for achieving a more comprehensive understanding of core concepts and skills in the two subjects. This paper focus on, how the educational relations between mathematics and biology at upper secondary education were strengthened through an intensive in-service teacher training program. We present the organization and structure of the program and an analysis of the teachers' reflections on Padlets with the aim of reporting about the teachers' conceptions of the challenges, limitations and potentials of interdisciplinary modeling activities between mathematics and biology.

Biology, mathematics and modeling

Throughout the history of biology, there are several examples of a fruitful and innovative interplay between mathematics and biology, and increasingly many biological phenomena are now described by mathematical models, while at the same time, biological ideas inspire new concepts in mathematical sciences. New experimental techniques in biology are producing enormous amounts of data, including full genetic and metabolic maps of organisms, and quantitative microscopy data that follow the locations of individual cells in organisms (Jungck 1997, Cohen 2004). The exceptional roles played by mathematics both in contemporary biology research and throughout the history have been unappreciated in biology and mathematics curricula as well.

In “International handbook of science education”, Berlin and White (1998) argue that science and mathematics are naturally and logically related in the real world, and educators therefore must try to capture this relationship in the classroom in an effort to improve students’ achievement and attitude in both disciplines

To reflect the interdisciplinary aspects of biology and mathematics, truly interdisciplinary teaching sequences are needed. Modeling activities take place in an interdisciplinary context and are therefore a possible frame for elucidation of the relations between mathematics and biology. The extensive literature recognizing the importance of models and modeling, both in mathematics education (Stillman, Blum & Biembengut 2015) and in science education (Gilbert 2004), and as an effective pedagogical strategy (Louca & Zacharia 2012), indicates that modeling might provide a generic methodology that can serve as a common ground for learning subjects such as biology and mathematics. Michelsen (2017) suggest a framework inspired by the notions of emergent model (Gravemeijer 1997) and model-eliciting-activities (Lesh & Doerr 2003) for coordination and mutual interaction between mathematics and the subjects of natural sciences building on two pillars (i) the conception of modeling as an interdisciplinary competency and (ii) a didactical model for coordination and mutual interaction between mathematics and the subjects of natural science. The didactical model consists of two phases: horizontal linking and vertical structuring. In the phase of horizontal linking thematic integration is used to connect concept and process skills of mathematics and natural sciences by modeling activities in an interdisciplinary context, e.g. modeling the process of consumption and removal of alcohol. The vertical structuring phase is characterized by a conceptual anchoring of the concepts, e.g. metabolism and concentration in biology and linear growth models, parameters and variables in mathematics, and process skills from the horizontal linking phase by creating languages and symbol systems that allow the students to move about logically and analytically within mathematics and the relevant subject(s), e.g. biology, of natural sciences without

reference back into the horizontal linking phase. The shift from the horizontal linking to the vertical structuring phase might thus concur with a shift from interdisciplinary teaching to discipline-oriented teaching. It should be stressed that the model is iterative. Once the concepts and skills are conceptually anchored in the respective disciplines, they can evolve in a new interdisciplinary context, as part of a horizontal linkage. Thus the underlying assumption of the model is that the disciplines are themselves the necessary precondition for and foundation for the interdisciplinary enterprise.

The “MathBio in the study package” program

In an attempt to offer upper secondary in-service teachers in mathematics and biology didactical tools to prepare themselves for the practical challenges of interdisciplinary teaching, the Laboratory of Coherent Teaching and Learning at University of Southern Denmark in collaboration with the organization Danish Science Gymnasiums in 2016-17 offered the professional development program “MathBio in the study package”. The overall aim of the program was to enable teachers to implement interdisciplinary teaching sequences between mathematics and biology in their daily classroom practices. The program involved 68 teachers from 20 upper secondary schools. It was the core idea of the program to involve teachers in design, implementation, and evaluation of innovative instructional sequences dealing with a wide range of aspects of mathematics and biology. The horizontal linking and vertical structuring framework presented above provided the teacher with a structure for developing the teaching sequences.

The program was organized as an intervention project structured around a combination of four seminars at the university and phases of practice at the participating teachers’ schools. The teachers were asked to work in pairs, one mathematics teacher and one biology teacher, and they participated in regularly meetings with mathematics and biology education researchers from the Laboratory of Coherent Teaching and Learning. The fundamental aim of the phases of practice was that the teachers designed and implemented an interdisciplinary mathematics-biology teaching sequence. At the seminars the teachers was introduced to the didactical model for linking mathematics and biology, different types of organizing interdisciplinary teaching, inquiry based teaching, examples of interdisciplinary mathematics-biology teaching sequences and presentations by researchers working on the interface between mathematics and biology, e.g. computational biology and reconstruction of body size by statistical methods. At the final seminar, the teachers presented their interdisciplinary mathematics-biology instructional sequences at a poster session.

In order to, get insight into the teachers' reflections and experiences regarding interdisciplinary teaching and the teachers' perceptions of constraints and potentials of implementing that the teachers themselves experienced two kinds of data were collected during the program. The teachers filled in a pre-designed protocol to keep track of the development of the teaching sequences during the program. The protocol included fields for teaching and learning goals, content of the teaching sequence, subject oriented and didactical reflections, evaluation of the teaching sequence and other issues relevant for the development of the sequences. As a part of the seminar programs the teachers were involved in discussing their views of the challenges, limitations and potentials of interdisciplinary modeling activities between mathematics and biology. The teachers used Padlet to report and share their discussions. The content of the Padlets were analysed to organize and thematically elicit meaning from the data collected and to draw realistic conclusions about the teachers' views of the challenges, limitations and potentials of interdisciplinary modeling activities between mathematics and biology. In the following we provide some preliminary findings from this analysis to shed light on the themes addresses by the teachers. The findings are illustrated by excerpts from the Padlets in italic and translated from Danish by the author.

Obstacles to, conditions for and ideal of interdisciplinary teaching

Halfway through the program the teachers in groups discussed obstacles to, conditions for and ideal of interdisciplinary mathematics-biology teaching and shared their views on a Padlet.

The teachers were first asked to identify the key obstacles to interdisciplinary mathematics-biology teaching. The analysis of the teachers' responses points on three main themes: (i) time, (ii) curriculum, and (iii) support from school management.

The teachers point at time as the main obstacle to a successful instructional interplay between mathematics and biology:

The high level of workload on teachers limits the opportunities for collaboration across the subjects.

Interdisciplinary teaching is according to the teachers time consuming, and it is crucial that adequate time is allocated to planning of and exchange of ideas about and experiences from mathematics biology interdisciplinary teaching.

Although interdisciplinary approaches are formally a central issue in the curriculum, the teachers see the curriculum as a major challenge to their efforts towards developing interdisciplinary teaching sequences:

Curriculum obstacles prevent cooperation between mathematics and biology when a particular type of assignment is to be considered for the exam.

The conceptual progressions differ from discipline to discipline.

The teachers emphasize the crucial role of the school management for a successful outcome of a professional development program:

The school management is not providing the necessary resources.

According to the teachers the school management should encourage teachers to pursue professional development, and ensure the time and the necessary physical facilities, and prevent problems with scheduling interdisciplinary lessons.

Subsequently the teachers were then asked to discuss the conditions for interdisciplinary mathematics-biology teaching, and the analysis of the responses crystallizes into three main themes: (i) interdisciplinary topics, (ii) common terminology and understanding, and (iii) the teachers' attitudes.

The curricula and the textbooks are still very discipline oriented, and there is among the teachers a demand for identifying interdisciplinary topics with a significant content for the participating disciplines:

Good interdisciplinary teaching materials.

Topics where both disciplines experience a central role are needed.

The different usage of the same concepts in the two disciplines leads the teachers to the insight, that a common language across the disciplines is an important condition for interdisciplinary teaching. The fact that teachers at upper secondary school are specialized in two disciplines, and therefore might have limited knowledge of the content and methods of other disciplines put a demand on the teachers to inform them on the other discipline:

Common understanding and language is a condition for interdisciplinary teaching.

Both teachers must be familiar with the basic concepts in the two disciplines.

The teacher colleagues' attitudes toward interdisciplinary teaching and collective professional development activities are also addressed as a condition for interdisciplinary teaching:

Openness and goodwill among the teachers – and coffee.

Views of interdisciplinary teaching from a disciplinary position

At a later instance the teachers were grouped by their subject and asked to reflect on how their own discipline benefits from and contributes to interdisciplinary mathematics-biology teaching, and to take critical view on interdisciplinary teaching. Again the teachers shared their views on a Padlet.

Both mathematics and biology teachers highlighted the benefits of interdisciplinary teaching for their own discipline. The mathematics teachers mainly focused on justification and application of mathematics:

Interdisciplinary modelling activities provide mathematics with a justification.

Mathematics is offered relevant topics for application.

According to the biology teachers biology offers real data, which can be processed by mathematical methods:

Biology offers more realistic data sets that challenge the mathematical models the students know from mathematics.

The teachers' responses to the question of how their own discipline contribute to interdisciplinary teaching show that the teachers consider the two disciplines as mutually supportive in the sense that biology delivers data, which are processed in mathematics:

Interdisciplinary mathematics-biology teaching offers the students a tool for handling and evaluating biological data. In some circumstances mathematical knowledge is necessary to draw conclusions in biology.

The idea of modelling as an interdisciplinary competence is adapted by the teachers. The teachers consider the biological experiment as an obvious common ground for the interplay between mathematics and biology:

In biology the students retrieve their own data and therefore they have a better understanding of why some data differs from the model.

Application of mathematics in biology could also support the transfer of knowledge between the two disciplines:

The interdisciplinary teaching sequences offer greater insight into the terminology of the two subjects and thus better opportunities to transfer knowledge between the two disciplines.

In continuation of this it is worth noticing that the teachers emphasize the necessity of reduction of the differences between the two disciplines:

The interaction between the two subjects requires that the subjects bend towards each other both temporally and in terms of content.

As teachers, we have through our participation in the program realized that the discipline matter of the two disciplines is not always the same, which makes it difficult for the students to transfer learning between the disciplines.

However, it should be noted that in spite of the teachers' positive attitudes towards interdisciplinary teaching they still accentuate the important role of the individual disciplines and a fear of confusing the students:

The interdisciplinary approach highlights the differences of the disciplines, and you become aware of the different approaches of the disciplines.

There is a risk of confusion and lack of overview and this might challenge the weak students.'

Concluding remarks

The "MathBio in the study package" program provided the teachers with the opportunity to get in on the ground floor of developing innovative interdisciplinary teaching sequences. The teachers creatively explored and forged interdisciplinary relationships between mathematics and biology. The experiences from the professional development program "MathBio in the study package" in the form of the teachers' reports on their development and implementation of the instructional sequences and the presentations given at the final seminar show that in general it is possible for the teachers from two disciplines to plan, carry through, evaluate and report about interdisciplinary modeling activities. The teachers gained insights regarding their teaching, in particular the limitations of the disciplinary approach and potential of interdisciplinary teaching. Across their disciplines the teachers supported each other in the development and the implementation of the mathematics-biology teaching sequences. The challenges faced by the teachers were curriculum constraints, missing time, differences in terminology between the two disciplines, fear of confusion, and lack of support from the school management. It is therefore of great importance to make the insight gained about interdisciplinary mathematics-biology teaching in the "MathBio in the study package" program available to a larger community of teachers and school managers. The major challenge is capacity building, which is providing support for teachers so that they can develop the understandings and skills required to teach for an interdisciplinary mathematics-biology curriculum.

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MODELS FOR INTERDISCIPLINARY TEACHING

Simon Zell¹

Abstract

Models for interdisciplinary teaching are a useful tool to describe and evaluate interdisciplinary teaching respectively interdisciplinary lessons. Furthermore they can be used for planning interdisciplinary lessons. Different approaches for models of interdisciplinary teaching are described and evaluated. This will lead to the design of an own model "Mathematics and Science under one roof". The potential and limits of this model will be described and discussed to show how this model can be used suitably by teachers and researchers.

Introduction

Teaching mathematics can be done in various ways. Teaching mathematics in an interdisciplinary context offers even more possibilities to plan math lessons. Therefore it cannot be said that there exists one way of interdisciplinary teaching (cf. Beckmann, 2003). She shows different aspects which characterize interdisciplinary mathematics lessons. Considering all of them, there could be more than 1000 different ways to teach interdisciplinary. There is no common view among researchers and teachers as well, what conditions should be fulfilled that a lesson is considered as interdisciplinary (cf. Zell, 2010, p. 39f). Quite a lot of papers agree with Jacobs (1989), that methodology and language from more than one discipline should be consciously applied (p. 8). Such an open understanding of interdisciplinary teaching is not shared by all. According to Fischer & Ruhloff (1993) interdisciplinary teaching is realized, if a common theme is taught in two or more subjects and will be the center of all teaching and learning. The planning of lessons needs to be coordinated and should be done in such a way that the subjects involved wouldn't give up its specificity (p. 122). They claim, that lessons should not considered interdisciplinary, if aspects of another subject are slightly touched or special methods like project work would appear. Lessons planned on the basis of such an understanding will be interdisciplinary for sure. On the other hand there many good realizations of interdisciplinary teaching would not be regarded as interdisciplinary. Therefore

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it is useful to follow a rather open definition like Jacobs (1989) while keeping in mind the critic of Fischer & Ruhloff (1993).

If evaluating interdisciplinary lessons or the interdisciplinary potential of lesson plans just by a definition of interdisciplinary teaching seems challenging. The evaluations would differ and difficult to compare. Quality criteria like objectivity and validity would be hard to fulfill. Models for interdisciplinary teaching can solve such problems and offer a categorization for interdisciplinary lessons. Using a model allows to operationalize such lessons making it easier to tell researchers and teachers if and to which extent a lesson can be seen as interdisciplinary.

Criteria on Models

Interdisciplinary elements in mathematics lessons should enrich disciplinary teaching (Beckmann, 2003, p. 30). They should lead to a deeper analysis of the concepts involved (Thurow, 1988, p. 49). These conditions should be recognized in a model of interdisciplinary teaching and consider the definitions and the rationales for interdisciplinary teaching. If such a model should be meaningful and useful for teachers it should give hints how realizable an interdisciplinary lesson plan would be. Reviewing literature for rationales of interdisciplinary teaching, Zell (2010) has setup four categories: holistic learning, constructivist approaches, problem-oriented learning and recognizing common structures and boundaries of subject-specific views (p. 45ff). Constructivist and problem-oriented approaches are not restricted to interdisciplinary lessons. Those can be realized in a disciplinary context as well. The main idea behind those rationales is that the realization would be more meaningful and realistic when taught interdisciplinary. The strongest arguments mentioned in literature belong to the category recognizing common structures and boundaries of subject-specific views. At least that category must be realized if using such models. Finally, evaluations done with a model should result to objective and valid results.

Models for interdisciplinary teaching

Different models will be described and evaluated according to the criteria mentioned above. The models which can be grasped the fastest are continuums models. The ends of the continuums represent disciplinary teaching. The more middle a teaching can be set, the more interdisciplinary it will be. The continuum model by Lonning & DeFranco (1997) should “characterize the nature of the relationship between Mathematics and Science being taught and the curricular goals for the disciplines” (p. 212). They distinguish five categories. Concepts which are best taught in a disciplinary context are classified to Independent Mathematics resp. Independent Science, which are situated at the ends of the

continuum. When contents of mathematics and science are both part of the curriculum for the same grade and instruction can be delivered in a meaningful way, then it is seen as balanced. If mathematics and science content doesn't belong to the same grade level it is classified as Mathematics resp. Science Focus depending on the content primarily taught. The primary target group of that model are teachers being involved in curriculum development to guide those choosing appropriate contents. A similar model is setup by Huntley (1998). There former discrete categories defined by the Education Development Center in 1969 have been transformed into continuous categories. Like the model of Lonning & DeFranco (1998) the ends represent separate approaches to teach mathematics resp. science. In contrast that model goes more into detail by looking how the interaction of both subjects takes place. The middle of the continuum represents lessons, where both subjects interact and support each other. "In this sense, there is more than just equal treatment of the two disciplines – there is a synergistic union of the two disciplines" (Huntley, 1998, p. 322). The other categories are described in more detail and looked from an observers' perspective. That model is also addressed for teachers reflecting their own teaching practice.

Continuum models can help understanding the extent of integration and clarifying the meaning of integration (Pang & Good, 2000, p. 75). Equal balance represents an ideal situation, which would never occur in real class (Lederman & Niess, 1998, p. 283f). One discipline will necessarily dominate the other one, since one subject is presented in another context, but still follows the conventions of the subject taught. Therefore they warn that attempts for an equal balance might result in approaches ignoring and dissolving disciplinary boundaries.

By applying the continuum models one can determine if a mathematic or science content can enrich disciplinary teaching and how a deeper analysis of involving concepts might be possible. The rationales for interdisciplinary teaching can be comprehended from the categories given. Since Huntley's model is more specific, more precise conclusions can be made, using her model. Both models give no clues how realizable interdisciplinary lessons could be. Only rough analyses can be made using these models and are not meaningful for deeper analysis of interdisciplinary lessons. By just looking at the place on the continuum it cannot be told, how integration takes place. Therefore more facts have to be written down, when evaluating an interdisciplinary lesson. Those might differ significantly, so objectivity and validity is not fulfilled or barely reached. For evaluating lesson plans they can just roughly tell the interdisciplinary potential. The model of Lonning & DeFranco (1997) relies on the curriculum. Hence the same lesson can be classified differently!

The Berlin-White Integrated Science and Mathematics model shall characterize and describe the integration of Mathematics and Science. According to their

creators it is the only model that “uniquely describes the center of the continuum, mathematics and science (Berlin & White, 2001, p. 52). It consists of six aspects: ways of learning, ways of knowing, content knowledge, process and thinking skills, attitudes and perceptions and teaching strategies. The three main aspects for interdisciplinary teaching are ways of knowing, content knowledge and process and thinking skills. The other aspects represent no distinguishing features concerning interdisciplinary teaching, but are often noted in literature. Along these six aspects current resources can be characterized and the development of new interdisciplinary teaching material be guided (Berlin & White, 2001, p. 53). The elaboration of those aspects will show its real value by identifying the links among the aspects and clarify the characteristics in defining integration. To assist the implementation of the model a checklist template has been developed to name relevant features of integration for each aspect but one, which are observable in integrated mathematics and science activities. These features are written as keywords like inferring, observing, communicating, classifying, deduction etc. Especially the keywords on content knowledge describe general features like change, conservation, patterns, symmetry etc.

All rationales for interdisciplinary teaching are touched by the Berlin-White Integrated Science and Mathematics model. Evaluation of interdisciplinary lessons can be done in a much more sophisticated way. By using the checklist objective results are possible and can be easily compared. But it is just telling if some aspect is touched and not how that would be realized. Therefore a checklist containing a lot of marks doesn't necessarily mean that one lesson is truly integrated. A checklist with a few marks may be much more integrated than one with many marks. If detailed evaluations beyond the checklist are carried out, then it will be more difficult to compare different ones for the same lesson, i.e. less objective. Using general keywords in the aspect content knowledge show commonalities between mathematics and science very well, but are unsuitable for evaluating an interdisciplinary lesson or a lesson plan. There the specific concepts and processes have to be investigated concerning commonalities and differences. Besides that, teachers planning definite concepts would not or only marginally be interested in general features.

Another model to show the different dimensions on interdisciplinary teaching has been developed by Labudde et al. (2005). It combines parameters of planning instructions with different levels of interdisciplinary teaching. They have come up with seven dimensions of interdisciplinary teaching which are based on literature and teachers' needs of their own research. These are categories, contents, interdisciplinary competences, teachers' role, methods of teaching, assessment and an additional one which can be filled out by oneself. Each dimension contains two till four facets and is illustrated by a mind map. The

center of that mind map is represented by the teacher and his subject, where he is socialized. Every branch represents a facet of a dimension. Each facet has several distinctions, the more distance is between distinction and center, the more a teacher moves away from disciplinary teaching. These will be described in more detail (cf. Labudde et al., 2005, p. 107ff): the dimension category describes how interdisciplinary teaching can be organized. The dimension contents don't contain specific contents but shows how it can be structured for interdisciplinary lessons. Interdisciplinary competences are not unique for interdisciplinary lessons. Though some of them can be learned more likely than in disciplinary lessons. Teachers' role is differentiated to the way of cooperation among them and their involvement, when teaching their interdisciplinary lesson. The dimension methods cover all general methods and should show that interdisciplinary teaching can be taught in various ways. The category Assessment regards the involvement of students in the assessment process and the format it is assessed. The last dimension is left empty on purpose to show that the model has not been completed and invites researchers and teachers to fill it out. The model shall show that interdisciplinary teaching can be done in various ways and gives hints how integration can be realized. The authors also state some limits of the model. More dimensions could be included like processes in science and some branches overlap. Comparison of lessons seem difficult, since a mind map structure might suggest, the more twigs are reached on the outer parts the more interdisciplinary a lesson would be. The intentional use of the model is a qualitative view of different types of interdisciplinary lesson and the interaction between the subjects (cf. Labudde, 2005, p. 112).

All rationales for interdisciplinary teaching named above are included in that model. This shows the deep connection between mathematics and science, since the primary intention of the model are natural sciences. The organizational forms of interdisciplinary teaching in the dimension category give clues on how disciplinary teaching will be enriched and a deeper analysis with concepts involved can be attained. The mind map contains 45 items on 16 branches, which is useful to show the variety of interdisciplinary teaching and can be a good guide in planning interdisciplinary lessons. For evaluating lessons and lesson plans that model is complex to use. Especially teachers have to be familiar with the meaning of some technical terms. Like the model of Berlin & White it doesn't include specific concepts and processes as needed for planning interdisciplinary lessons. Recognizing the common structure and boundaries of subject-specific views cannot be read out of the model directly, but must be derived from general terms given. It also needs some time to get clues how realizable an interdisciplinary lesson or lesson plan would be.

All these models can be applied to investigate single lessons or lesson plans. But when doing that all the models lack on their specificity, since they all deal with general features. They ought to be derived to a special case. Besides that, all models seem not to be objective, i.e. the conclusions done by two persons might differ significantly. A model describing the extent of interdisciplinary teaching should take these aspects into consideration.

The model "Mathematics and Science under one roof" will close those gaps. The principal structure can be seen in Figure 1. It consists of three main aspects: content, heuristic competences and organization. In the upper part of each column the appearing aspects of a lesson (plan) are written there. They are evaluated in the lower part, where it is written down how each one is used resp. intended to use in class. At the very top a short description of a lesson is given. So by looking at the top of the model, one can get a quick overview about the interdisciplinary lesson. This is intended for teachers planning a lesson to a given topic. It will make them easier to decide if that lesson would be suitable for their teaching without going into detail. The lower part of the model is the main part for evaluating interdisciplinary lessons. At the very bottom there is space for further items which can't be assigned to the three categories but are significant for the evaluation.

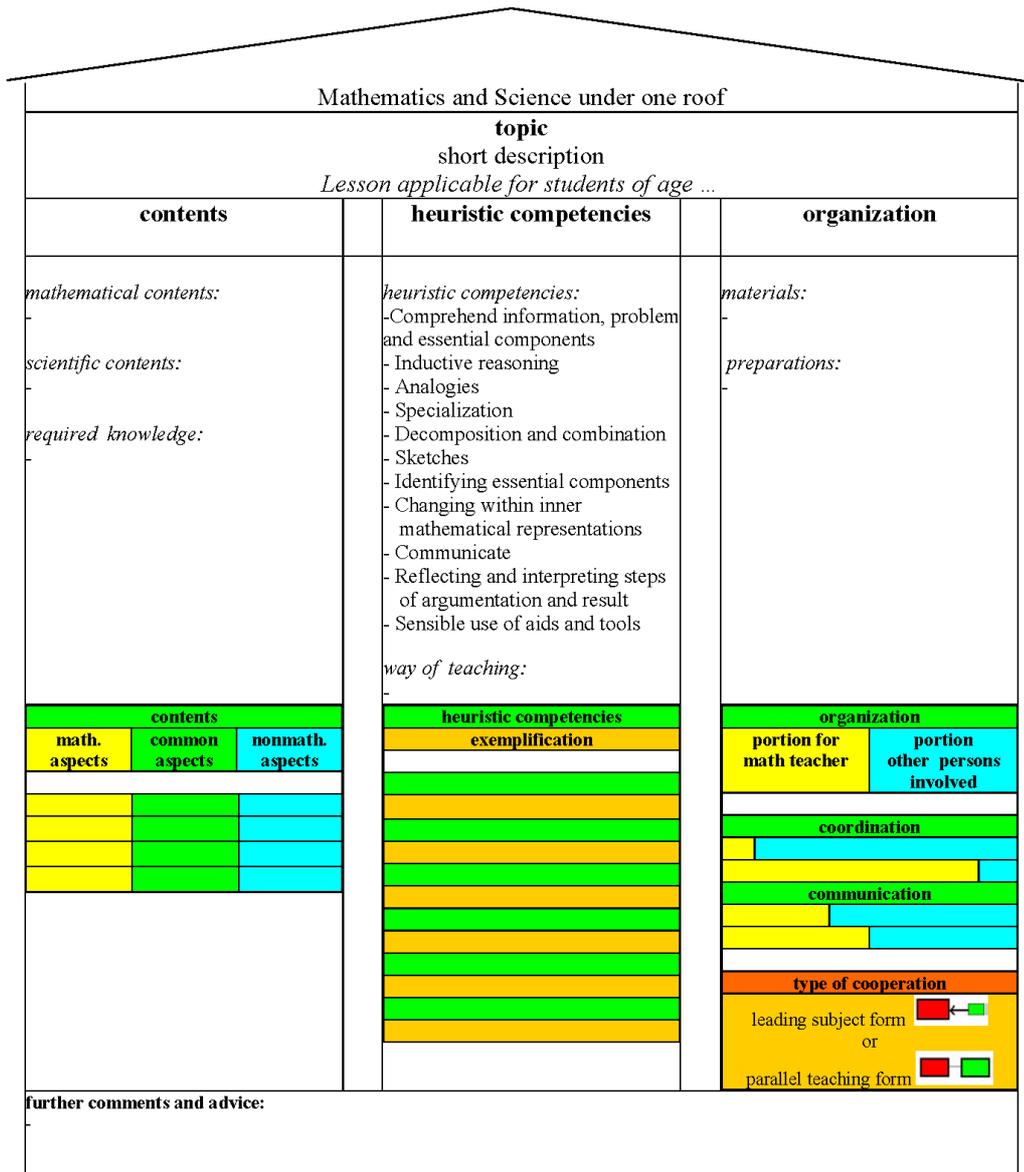


Figure 1. Mathematics and Science under one roof.

The three categories are now described in more detail. Contents touching mathematics and science are often the basis for interdisciplinary teaching. By looking at the content of a lesson, one can best differentiate between common and disciplinary aspects/views of the involving disciplines. So common structures and boundaries of disciplinary views can be illustrated. By looking at that lower part of the content category the realization of a holistic interdisciplinary lesson can be recognized. It can be filled out for a mathematical concept or aspects of mathematical concepts. The second category heuristic competences sets the focus on process skills and the way how mathematical and scientific concepts are used. In figure 1 the main heuristic competencies are listed acknowledging that other competences may be added. These ones were the dominant ones, when validating the model. In contrast to the content category

the heuristic competences are not differentiated into common and disciplinary views, since there the commonalities dominate. When testing the model on several lesson plans, it has been observed that adding disciplinary views hasn't been suitable. The category organization primarily deals with the aspects coordination and communication. These two aspects are the major factors which influence the realization of interdisciplinary teaching (cf. Zell, 2010, p. 73ff). To roughly show the parts, each involving teacher needs for planning, the rows underneath coordination resp. communication are meant to be flexible, taking into consideration that the need for cooperation resp. communication depends on the individual teacher. At the bottom the type of cooperation between the subjects is intended. The leading subject and parallel teaching form are according to Beckmann (2003) and show how the involving disciplines interact with each other. These two mentioned are the most realized ones in class (cf. Zell, 2010, p. 73ff). The third category is the main category to show how realizable an interdisciplinary lesson can be.

The model is designed as a tool for teachers and researchers to describe how interdisciplinary mathematics and science lessons can be realized. It also serves as a planning tool for designing interdisciplinary lessons.

Besides the rationale recognizing common structures and boundaries of disciplinary views, the other two rationales problem-oriented learning and constructivist learning can be touched by the model. By examining the heuristic competences, the extent of problem orientation can be read out. Looking at the aspects in the content category gives information what kind of perceptions can be used in that lesson and show the potential for constructivist approaches. Both categories, content and heuristic competences can show how the learning of mathematics and science can be enriched by an interdisciplinary approach. Especially by looking at common and different aspects of the concepts one can tell how a deeper analysis of the concepts involved can be reached. It can also give information how realizable a lesson plan would be. Since the model gives a structure that keeps the balance of being too strict or too open it offers flexibility for the users without losing too much objectivity. It has been validated on 26 interdisciplinary lesson plans involving mathematics and science.

Exemplification of the designed model

The use of the model is illustrated on a lesson to use a buoyancy experiment in a mathematics lesson to discover aspects of the concept of variable. Beginning with an impulse from everyday life, the students are getting familiar with the phenomena and can make connections to the experiment they are to do in the following. After measuring the force of single weights in air and water, they have to measure systematically and have to find a formula which fit to their measuring

values. For their variables chosen they have to determine their changing behavior and all values possible. Then they shall make hypotheses how the formula changes when salt water is used instead and come up with a new formula. By doing that, they discover that variables can represent both continuous and discrete values and explore the changing behavior of variable on a definite example. Using experiments to discover the concept of variable have great potential to discover those concepts and enrich authentic mathematical learning (cf. Zell, 2009). A possible evaluation of that lesson is shown in figure 2.

Mathematics and Science under one roof																																																																	
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further comments and advice: - at task 5, help students finding formula, if necessary - combination with Boyle's law and thermal expansion experiment possible, if not enough material for whole class - if students have no experience in handling measuring errors, give short introduction into that topic in advance and remind students that measuring errors appear when they try to find the formula. - main emphasis on mathematical aspects; discussing physical aspects after that lesson or in physics class if one wants to do so - teaching material prepared as leading subject form, can be modified easily to parallel teaching form																																																																	

Figure 2

It shows that the concept of variable is variously touched. Variables are represented as generalized number, specific unknown and in a functional relationship. The common and different views of mathematics and physics can be illustrated: from a mathematical perspective variables are context free quantities whereas in physics certain letters indicate a physical quantity and are referred to a concrete experiment. These different perceptions can be observed in the following rows as well. Just by looking at the first category it is possible to see the great potential for interdisciplinary teaching. But it also shows that teachers have to be aware of these perspectives which can influence students' learning.

The category heuristic competencies show that many can be touched by such an approach. Reading the exemplification is helping teachers to be better aware how these competencies are used. The third category gives a good overview about the work to be done before teaching that lesson. It shows that the time for coordination and communication is higher than a regular mathematics lesson. Looking at the interdisciplinary potential, showed in the first category, that extra time is worth to invest.

The further comments at the bottom give information about students' difficulties which might occur and possible modifications on the lesson plan to intensify interdisciplinary learning. The example shows: using that model helps teachers and researchers to see the interdisciplinary potential of a lesson plan quickly. Its structure offers a quick overview of a lesson (plan) and also detailed investigation of the major aspects of interdisciplinary teaching.

The application of the model is most suitable for single lessons which cover one major concept. Using it for several interdisciplinary lessons covering different contents would make the illustration too complex. Then it would be harder to find out the commonalities between both subjects. In that case it would be better to apply the model several times.

Filling out the model takes time. Therefore, it is not the intention that teachers should fill out the model in addition to planning their lesson. But looking at the template (Fig. 1) helps teachers to get hold of the major aspects of an interdisciplinary lesson. For persons developing interdisciplinary teaching material a model filled out can be a useful additive to give a quick overview of the material and show the interdisciplinary potential.

Conclusion

Models for interdisciplinary teaching facilitate to show the interdisciplinary potential of interdisciplinary lessons between mathematics and science. Continuum models may give a quick result but cannot give more detailed

information by the continuum itself. The models by Berlin & White (2001) and Labudde et al. (2005) illustrate the major aspects of interdisciplinary teaching and how various interdisciplinary teaching can be. But they are less suitable for single lessons since they concentrate on general features. The model Mathematics and Science under one roof allows both showing main features of interdisciplinary teaching and realization for single lessons. Therefore it is more suitable for teachers planning interdisciplinary lessons. By the structure given, objective results of lessons evaluated are more realizable than it would be possible using the other models. So Mathematics and Science under one roof appears to be more suitable for researchers as well. It is the model which illustrates best how realizable an interdisciplinary lesson can be.

Further research may show how the model can be generalized to interdisciplinary teaching involving more than two subjects. Since the model of Labudde et al. (2005) is primarily intended for interdisciplinary science lessons and suits for interdisciplinary lesson between mathematics and science as well, it could be worth investigating if the model Mathematics and Science under one roof can be also applied for interdisciplinary science lessons.

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STEAMING SOFT-SKILLS IN MAKERSPACES: WHAT ARE THE MATHEMATICAL CONNECTIONS?

Manon LeBlanc¹ · Viktor Freiman¹ · Caitlin Furlong¹

Abstract

Soft skills are now considered an important part of digital competencies in a lifelong continuum. As continuation of our work on the CompÉTICA (Compétences en TIC en Atlantique, ICT Competencies in Atlantic Canada, www.competi.ca) project presented at the MACAS-4 Symposium in 2015, we continued researching digital competencies while observing exemplary practices in different contexts through lifelong learning paths. One of such practices, in constant expansion in many countries, is learning in school makerspaces where students choose, plan, design, and conduct all kinds of multidisciplinary projects in the areas of STEAM (Science, Technology, Engineering, Arts and Mathematics) education.

At the end of the 2015-2016 school year, the CompÉTICA team conducted observations in six local schools where makerspaces were created over the past 2 years. In 2016-2017, four of these schools took part in an in-depth study with a focus on the development of soft-skills among students working on their projects over the whole school year. During the Symposium, we will reflect on the first results from this case study while looking for emerging mathematical connections.

Introduction: digital skills and issues of STEAM education

In Canada, as elsewhere in the world, the government is multiplying its efforts to define, develop, and foster digital citizenship. To become a digital citizen, one needs to develop “new” abilities and skills in order to have lifelong success in his or her education, career, community and family life. These skills refer, among others, to the ability to solve problems in technology-rich environments, and more precisely to the use of “digital technology, communication tools, and networks to acquire and evaluate information, communicate with others, and perform practical tasks” (OECD, 2012, p. 47).

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Three core dimensions of problem solving in technology-rich environments have been identified by the OECD: the task or problem statement, the technologies, and the cognitive dimensions. The task refers to the situation that will get the learner engaged in understanding the problem at hand. The technologies encompass the devices, applications and functionalities used to solve the problem. Finally, the cognitive dimensions are what separate problem solving in technology-rich environments from ICT competence, which has to do with the use of technical skills. The cognitive dimensions go beyond the simple use of digital technologies and refer to the mental structures and processes one uses to find a solution to a problem.

In the past years, a new trend has made its way all across Canada: the STEAM movement (S=Science, T=Technology, E=Engineering, A=Arts and M=Mathematics). A national initiative, called “2067 Canada – The science of a successful tomorrow”, aims to shape the future of science, technology, engineering and math learning, focusing on K 12 students (Let’s Talk Science, 2017). Originally framed as STEM education, a federal “Canada 2067” initiative will develop an action plan and a national vision that will ensure young Canadians are prepared to compete, thrive and contribute in the rapidly changing world. In an article of the national Globe and Mail newspaper (October 21st, 2016), the president of Universities Canada, Paul Davidson, vows for an increasing role of liberal arts and humanities, even in engineering degrees, thus emphasizing interdisciplinary connections in higher education. In the context of educational makerspaces, these connections are particularly prominent; hence, we have been using the acronym STEAM as a reference for our work within the MACAS Symposia since 2005.

The need to foster digital citizenship and thus develop the skills associated with problem solving in technology-rich environment is generally accepted. Despite this, however, the way that these skills are structured and the modelling of their acquisition at every stage of life is still in full development (both from a research point of view and from a practice point of view). How are they developed in schools? In this study, we looked at exemplary practices and more precisely at makerspaces and the place of mathematics inside these creative laboratories.

Context of our study: digital skills in Atlantic Canada

Technologies have been part of the educational landscape in Atlantic Canada for over 15 years. However, despite the many efforts made to help teachers integrate technologies in their classroom, there is a need for more training and personal development amongst teaching staffs (Freiman & al., 2015). The transdisciplinary outcomes for K-12 students found in every school curriculum are also outdated. They mainly focus on instrumental skills and the use of digital

technologies and are quite far from what is expected from students nowadays (Freiman & al., 2016). This could explain, at least in part, why a study carried out in 2012 by OCDE PIAAC shows a lack of student competence in problem-solving in technology-rich environments (Statistics Canada, 2013).

To better understand digital competencies (including problem solving in a technology-rich environment) and more precisely to identify and measure the acquisition and transfer of lifelong digital skills, a partnership was created between actors from different backgrounds (school, university, business, etc.). This research partnership network, named CompÉTICA (Compétences en TIC en Atlantique, which translates as ICT Competences in Atlantic Canada – www.competi.ca), aims to establish collaboration between a) family and school; b) elementary school and high school; c) high school and post-secondary institutions; d) educational institutions and the workplace. Its main objectives are to 1) define the continuum of digital competences in varied contexts of life such as education, work, community and family, 2) identify and describe best practices that develop these competences over a lifelong period, and finally 3) develop and implement new initiatives as the result of increasing collaborative efforts (Media Awareness Network, 2010).

In order to develop the continuum of digital competences, in the first year of the project, 40 experts were interviewed. We used MediaSmarts' theoretical framework on digital literacy to study the best practices identified in the various environments associated with different stages of life. Firstly, the model highlights the fact that without access, it is futile to talk about digital literacy. Three levels of digital literacy are then presented hierarchically (from the least to the most complex): Using, Understanding and Creating. The results that emerged from the data analysis show that to be able to efficiently and productively use digital tools and environments, four main elements are required: 1) Access to the technology; 2) Instrumental skills; 3) Research skills; 4) Creation and communication with ICT (including social media).

Moreover, a higher level of competences requires a combination of instrumental skills and soft skills. Among the latter, we find creativity (including the possibility to publish online), critical thinking, autonomy and intuition (Freiman & al., 2015; 2016). The mere possession of these skills does not guarantee that a person will become a digital citizen. Indeed, it is the combination of these skills and the ability to transfer them from one context to another that is essential (LeBlanc & al., 2016).

The STEAM movement has also reached Atlantic Canada. In New-Brunswick, within the last two or three years, several makerspaces (also called fab labs,

creative labs or STEAM labs) have been created. Such labs aim to create a new society where technology plays an important role in problem-solving (Willson, 2016). According to Rendina (2015), a creative laboratory is a place where students can make use of a variety of diverse materials to discover, invent, explore and create in an environment that encourages the construction of their own learning. Therefore, in makerspaces, students learn differently. They undertake a project (mostly creating objects of all kinds, useful or for fun) by first exploring, then integrating technology, and finally by being technologically creative (they show inspiration and innovation) (Willson, 2016).

Thus, these labs provide an environment in which students can design, experiment, build and invent, while learning about STEAM. Activities can go from cardboard construction to electronics, programming, robotics, and sewing. These informal activities, although previously known, took a new life in the early 2000s with the advent of ICT. It seems obvious that all these activities involve science, technology, engineering and arts. But where are the mathematics? And more importantly, what kind of mathematics does the “M” stand for in STEAM? We know that there are three kinds of mathematics: the curriculum mathematics (associated with learning outcomes), the « real-world » mathematics (for instance, deciding whether to invest some money in a low risk or high risk stock) and the « new » mathematics (emerging from the activity and not the curriculum). Sadly, the curriculum mainly focusses on disciplines and does not necessarily reflect the trend of transdisciplinarity put forth by the STEAM movement. Can the development of soft skills facilitate a more integrated curriculum (with a more holistic view of education), in respect to the MACAS legacy?

Theoretical framework

Soft Skills

Soft skills are defined by Hurrell, Scholarios and Thompson (2013) as skills that are “non technical and not reliant on abstract reasoning, involving interpersonal and intrapersonal abilities to facilitate mastered performance in particular social contexts” (p. 162). They include oral communication, team work, customer handling and self presentation. Other authors, like Wheeler (2016), consider emotional intelligence as a soft skill. In such a case, we then refer to self-awareness, conscientiousness, adaptability, critical thinking, attitude, initiative, empathy, self-control, organizational awareness, leadership, time management, political savvy, likability, and persuasive ability.

Other researches on soft skills have focussed on the transition from one stage of life to the other. For instance, Crawford and al. (2011) analysed the results of a

cross institutional survey that was given to students, faculty, alumni, and employers in order to pinpoint the soft skills that are needed to make a smooth transition from university to a career. The results show seven clusters of soft skills that seem to be important to employers. They are, in order of importance, communication, self-management, teamwork, decision-making and problem-solving, experiences, professionalism and leadership.

No matter what the context is, authors usually agree that soft skills are cross-sectional competences that are an expression of how people know and manage themselves and their relationships. They are therefore essential to function in society.

Makerspaces: new learning spaces for STEAM education

Makerspaces are known as learning spaces that present different layouts, in which students are engaged in a multitude of projects, during which they share and explore with others while working with various technologies (Brilliant Labs, 2016). Learning is seen both as an autonomous activity (self-directed learning, problem solving) and a networking activity (self-determined learning, problem posing, collaboration with others to build together a path of investigation). In this pedagogy, the teacher is first and foremost a guide or a facilitator, with her or his main focus being to accompany students in a culture of collaboration and curiosity (Gerstein, 2016).

Elements of this pedagogy can be retraced in history. Indeed, the works of Piaget, Vygotsky, Papert, Lave and Wenger have influenced the STEAM movement. Martin (2015) cites Montessori (1912) and his ideas of bringing children to learn by building with interesting tools and materials. The tools have evolved, but the big idea is the same.

Several possible learning benefits are mentioned in the literature. Interdisciplinarity is often found to be one of the main advantages of school makerspaces that have a particular focus on STEAM skills (Litts, 2015). Sheridan, Halverson, Litts, Brahms and Jacobs Priebe (2014) argue that the work in makerspaces fosters students' autonomy and collaboration. Other elements mentioned in the literature are the development of critical thinking and argumentative skills (Litts, 2015), the increase in the capacity to do problem-solving, as well as the use of gaming in learning, which can make learning more cooperative (Connected Learning, 2015).

Overall, the authors seem to agree that when working on their projects, students explore different possibilities for their future career choices (Litts, 2015), while becoming active members of their learning community (Sheridan, Halverson, Litts, Brahms & Jacobs-Priebe, 2014). Moreover, this experience seems to engage

and motivate young learners because they do work according to their personal interests (Litts, 2015). Some researchers have also observed increased perseverance and self-determination from the part of learners (Yeager & Dweck, 2012).

Methodology

Various case studies, each focussing on a particular theme, were conducted throughout a period of two years. Here are some of the issues that were discussed: a school turning to ICT integration (Léger & Freiman, 2016); the teachers' perspectives of digital literacy learning outcomes; programming and robotics (Djambong & Freiman, 2016); intergenerational families (Godin et al., 2016); postsecondary education (university, community college); informational literacy; musical creativity with iPads (Robichaud & Freiman, 2016); makerspaces. The preliminary results from this last case study (makerspaces), where we looked at the development of soft skills and the place of mathematics in STEAM-based education, are the main focus of this text.

We did an exploratory descriptive study that took place from 2015 to 2017. In the 2015-2016 school year, six school makerspaces took part in the study. Each of these makerspaces was visited once at the end of the school year (in May or June). At this stage, the goal was to take an initial look into the new school-based learning spaces and describe how the makerspace was organised, what kind of projects were running and how the students (n=180) and teachers (n=12) perceived their experience. In 2016-2017, we continued the study in four of these six makerspaces by visiting two or three times during the school year (in November, March and June). Our main focus was on the skills students were acquiring while working on their projects over the school year. We made video observations of students while they were working, asking them what they were doing, why they were doing it, and how they were doing it. We also tried to take notice of the students' progress from one visit to another by asking them how their project developed during the past months, what challenges they met and how they responded to them.

While the whole study was aimed at the diversity of skills students were developing, for this paper, we focus specifically on mathematical connections with other subjects in a makerspace environment. We selected a sample of five videos, totaling 51 minutes and 26 seconds, which were each analysed by two researchers. We looked for instances where students were either explicitly talking about mathematics or doing some mathematics without necessarily noticing it. Essentially, we wanted to identify the mathematics that students were doing while working in the makerspaces. The two analyses for each video were then compared, in order to ensure intercoder agreement validation.

Results

Some tasks seem to facilitate mathematical connections with others subjects more than others. Amongst those tasks, we find robotics (which includes building and programming), constructing models (in our case, a model of the Acadian flag), 3D-Printing, improving an already existing object (for example, modifying handlebars in order for them to be heated for wintertime), story writing (e-book) and building electric circuits. It was clear to us that mathematics were highly involved in all of those projects. Here is a more detailed description of the observed mathematics.

We observed two types of mathematics: the “obvious” mathematics and the “hidden gem” mathematics. The mathematics in the first category are pretty basic mathematics and serve as a tool. They are what you would find in any curriculum. For example, when they were building their models, students were measuring and comparing heights, talking about proportions and fractions, locating objects in a plane, etc. When they were building the heated handlebars or electric circuits, they were working with the four basic operations and talking about different units of measure (converting centimeters to meters or vice versa, working with volts, amps, minutes, etc.). We also saw students estimating (for example, estimating the amount of time it would take to read a paragraph for the story). The mathematics in the second category may be unnoticeable to the untrained eye, but in our opinion, they are the mathematics that show real understanding from the students. They are mathematics for meaning. They have been observed in almost all of the projects in the makerspace. Table 1 shows a list of those “hidden gem” mathematics and translated excerpts of students’ words from the videos.

Mathematics	Excerpts from the students
PIE strategy (Prediction, Investigation, Explanation)	“We used a USBC, then we cut it. Then there were 4 wires. There was the white one that I cut, the yellow one that I cut because it was not them ... the red and the black, they are the ones we use. In total, they have five volts, then when it passes through it divides. That's why we need a big battery so that it can charge [...] It will not last five seconds, it will last 1h, 2h. We have not tested how long it was.”
Problem-solving (during a “long” period of time)	“It's to warm the handlebars like on your bike so that in the winter you will not have a frostbite.”

(In)validation	<p>“Right now we're measuring the volts then the amps to see if we can have a switch. [...] We had measured with just one, now we're trying with both.”</p> <p>“I make a lot of mistakes and I learn from these mistakes.”</p>
Logical relations	<p>“The pig, if I had not used numbers and measuring, it would have been too big, then it would have taken like 2 hours to do everything.” (relation between the width of the pig and the time required to print in 3D)</p>
Inductive reasoning	<p>“Then there, equal, then after that the answer” (referring to programming and linking what they had done before to what they had to do now and next)</p>
Creating a plan	<p>“We are writing a story with classes from all over the world. We do it with PowToon, so we make a character move...” (while explaining, the student shows the plan for the story)</p>
Respecting constraints	<p>“She has less than a minute to read one of the paragraphs. She has a minute max.”</p>
Management of variables	<p>“The program was already done, but we changed a few things. We've changed the tone there...” (explaining how values were changed, in order to achieve the desired results)</p>

Table 1: “Hidden gem” mathematics observed in the makerspaces and excerpts from the students

Conclusions and discussion

What is the role of mathematics in STEAM? In our analysis, we observed two types of mathematics: mathematics as a tool and mathematics for meaning. Both are essential to the full development of students. In fact, both are fundamental to the development of soft skills. The “obvious” mathematics represent the technical skills, whereas the “hidden gem” mathematics are more

transdisciplinary. Without “obvious” mathematics, students can’t problem solve, see relations between concepts or predict, investigate and explain what could happen in a given situation. On the other hand, without the “hidden gem” mathematics, technical skills remain exactly that.

After this experimentation, we find ourselves with more questions than answers. Is there enough technology in our schools to develop soft skills in students (in mathematics and elsewhere)? Is stowage between technology and the curriculum a problem or a welcomed challenge? If we are up to the challenge, are we sure that the time and money invested in logistics and coaching are worth the effort? How do we know if it is worth it? If we are doing all of this to help students function better in tomorrow’s society, how can we assess the benefits or misdeeds of the integration of technology in our schools? This last question brings up difficulties from a methodological point of view. Indeed, how can we capture an action in makerspace, when the environment is so chaotic?

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TECHNOLOGY-BASED GAMES IN MATHEMATICS AND THEIR IMPACT ON STUDENT MOTIVATION

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Abstract

At the secondary level (high school), there appears to be a lack of student motivation in mathematics courses (Wigfield, 1994, cited by Chouinard, 2001), and this lack of motivation can have negative repercussions on student learning and performance. Some authors argue that integrating games in the classroom is a good way of increasing young people's interest in mathematics (Sauvé, Renaud and Gauvin, 2007). My thesis aims to verify if high school students are more motivated to learn in their mathematics courses when technology-based games are integrated within the classroom. This project, still in progress, is a case study. A series of technology-based games (that deal with the mathematics curriculum), such as computer games, games where the teacher uses technologies, action games, and so on were proposed to an 11th grade mathematics teacher and his students. During a five-month period, these games are being integrated into the regular classes and the students are observed using a checklist, to verify that motivation indicators are present when the students are playing the games. The results that will be the subject of this presentation allow us to observe the fact that during the integration of these games, students are having fun, are engaged, are focused on the task for a longer period, are demonstrating mathematical abilities and are motivated to go beyond the scope of their mathematics class, i.e. exploring functions they haven't seen yet.

Literature Review and Problem

Lack of motivation in math classrooms

There is a link between motivation and learning. Indeed, according to Leblond, "school motivation influences students' achievement and the strategies used to achieve it" (2012, p. 14, free translation). In other words, motivation can affect the amount of effort a student puts towards learning. Sadly, the lack of motivation for academic studies is prevalent among today's young people, especially when

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it comes to high school students, and for this reason, they have no desire to fulfill the tasks required at school (Vert -Demers & Pelletier, 2003, cited by Legault, Green-Demers and Pelletier, 2006). According to Snyder and Hoffman (2002) and Statistics Canada (2002), a large proportion of high school students lack motivation (cited by Legault, Green-Demers and Pelletier, 2006). There also seems to be a decline in student motivation towards mathematics, and as stated by Leblond (2012) this “corresponds to a disengagement of learning related to this field and of course the careers associated with it” (p. 15, free translation). In other words, a student can be interested in a particular field of study, but if he or she knows that math is involved with this field, they will disregard it.

Moreover, many students do not have the required mathematical skills to function well in society (Dossey & al., 1988, cited by Meyer, Turner, and Spencer, 1997). We can see this on a daily basis where people cannot do simple additions without the help of their calculators. This shows how important it is for students to be motivated to learn mathematics in high school.

Games as a motivational tool

One aspect that can motivate students is games. Sauvé, Renaud and Gauvin (2007) explain that games “positively support self-esteem and self-confidence, commitment, a desire to persevere and accomplish the task” (free translation, p.100). They allow students to not be afraid to make mistakes and permit rapid feedback. Unfortunately, games are often associated with primary education, so they are less commonly used in high school.

Since games are a motivation tool, it is important to understand what a game is. A game is “an artificial situation where players are placed in a position of conflict with one another, or together against other forces, and are governed by rules that structure their actions for a specific purpose, to win, to be victorious or to take revenge “(Sauvé, Renaud & Gauvin, 2007, p.95, free translation). It is important to note that simply integrating a game in the classroom is not enough to ensure student motivation. As stated by Meel (2002), Winograd (2001), and Moyer and Boylard (2003), “learners must feel actively involved and challenged in the course of the game, which provokes a desire to persevere, accomplish the task “ (cited by Sauvé, Renaud & Gauvin, 2007, p. 97, free translation). This becomes particularly tricky when the students are placed in groups which allows some students to hide. For this reason, it is important for teachers to have good classroom management to assure that everyone participates.

Technology as a motivational tool

Another element that motivates students is technology, mainly because they are familiar with it, and it allows a better level of student engagement as well as a better understanding of the material being studied (Muir-Herzig, 2004). In this day and age, there are many technological resources that can be used in high schools, however, not many high school teachers use these technological resources to their full potential to help motivate students (Busana, 2001).

Objective

Because games and technology motivate students, the objective of this research is to examine whether young people will be more motivated to learn high school mathematics if technology-based instructional games are integrated into classroom activities.

Theoretical Framework

Motivation indicators

Viau (2004) points out four main characteristics that can be associated with a motivated person: perseverance, choice, cognitive engagement and success (performance). Success will not be taken as an indicator of motivation in this research, because to measure performance, quantitative analyses would have to be done, and this is not part of the study's objective. In the context of research on the impact of ICT on motivation, interest and pleasure also emerged as indicators of motivation (Viau, 2005). These motivation indicators will be explained in the following paragraphs.

Perseverance: "Perseverance is observed by the sufficiently important time that the pupil devotes to his school activities to enable him to perform them well" (Lacroix and Potvin, p.3, free translation). In other words, if the student remains on task, even in difficult times, we can see that he perseveres. A student who perseveres will usually maintain a positive attitude towards the task he or she must perform.

Choice: Secondly, there is the choice to participate. "A student who participates is a student who listens in the classroom, actively participates in the various activities, asks questions to better understand or deepen a notion; It is a pupil who does the exercises, the readings and the works required "(Barbeau, 1993, p.11, free translation). A student who does not want to participate will have avoidance behaviors. For example, during an activity, a student that does not want to participate can be found asking unnecessary questions, being off task or

simply refusing to do the activity. If a student does not present this avoidance behavior, it can be concluded that he or she has chosen to participate.

Cognitive engagement: Cognitive engagement refers to the degree of mental effort that the student deploys when conducting an educational activity (Salomon, 1983). If a student uses learning strategies to successfully complete a task, it is possible to see that he is engaged.

Interest and Pleasure: Intrinsic motivation reflects behaviour that is undertaken for its own good, enjoyment and interest with a high degree of perceived internal control. The components of intrinsic motivation are interest and pleasure. Self-interest refers to a person's willingness to be engaged in a particular activity, such as mathematics (Pintrich, 2003), while situational interest represents the physiological state of a person to be attracted to a certain situation (Pintrich and Schunk, 2002, cited by Pintrich 2003). Pleasure is "characterized by the search for a positive emotional state" (Schillings, 2003: 15), so a young person who is experiencing pleasure will have a positive attitude, feel joy and say that he or she likes what he does.

Categories of games

There are many different categories of games that exist. De Grandmont (1997) proposes three main categories to define game based learning: ludic games, educational games and pedagogical games. Ludic games are based on the notion of fun and development. These types of games are usually used with younger children or as an ice breaker for older children for fun and for developing social skills. Educational games are learning oriented, but not in line with the learning outcomes. Pedagogical games call for knowledge and are in line with learning outcomes. For example, if a game is about sciences, but doesn't coincide with the learning outcomes of the classroom, the game would be educational, but not pedagogical.

There are also three important sub-categories of games which are "off the shelf" games, pedagogical games created by educators or professionals and pedagogical games created by students. First, the "off the shelf" games are often associated with the educational games category. This type of game involves the integration of existing games into the classroom without them having necessarily been designed as pedagogical games (Van Eck, 2006). It is less recommended to integrate these games into the classroom, because it is difficult to find a game at a store that directly relates to learning outcomes.

The next game sub-category is educational games designed by educators or professionals. As the name suggests, these games are created by educators or programmers and are developed according to a pedagogical objective (Van Eck,

2006). Thus, they are designed to be integrated in the classroom or at home for educational purposes. The only disadvantage with the use of educational games that are already created is that some of them are expensive, but there are often discounts for teachers. Moreover, with the growing popularity of ICT, one can find many of these types of games online, but again, they are not all free.

Pedagogical games created by students are often technological games. "The construction of games puts children under control of their own learning and reflection and leads them to plan and manage the complex process of creating a game" (Kafai, 1995, cited by Vos, Van der Meijden and Denessen, 2010, free translation, 127). Increasingly in schools, young people learn to program, and students create their own games. On the other hand, if young people do not know how to program, the process of creating technological games becomes more difficult. It is therefore rarer to see this type of game in the classroom.

There are many other subcategories of games, but for this project, we will be looking at technological games and games that have the possibility of being played with the help of technology. This category of technological games is a big category, because there are a lot of technological games. The description is simple, these are games that are played with the help of technologies. Since technologies are constantly evolving, this category has great potential for the future. If you divide this category into sections, you could find, for example, video games, computer games, and online role-playing games.

There are other types of games that can be played with or without the help of technology. One of these types of games is adventure games. These games consist of a narrative environment where students must test hypotheses and solve problems (Sauvé, Renaud & Gauvin, 2007). Several video games are in this format, such as *Mario Bros* games, for example. When adventure games are developed for educational purposes they can be exogenous or endogenous fantasy games. Exogenous fantasy games, also called *edutainment* games, are where game principles are simply superimposed with academic content (Buckingham & Scanlon 2003, Egenfeldt-Nielsen, 2005, Okan, 2003, cited by Delacruz, 2011). The intention is to motivate young people to accomplish tasks by playing a game where the academic content being studied is integrated into the game, but it is not intrinsic to it (Kebritchi, Hirumi, & Bai, 2010, cited by Delacruz, 2011). In other words, the students play the game and at some point have to solve a math problem to continue the game. In an endogenous fantasy game, players learn and practice skills to accomplish game objectives, and this makes learning meaningful and intrinsically motivating (Gee, 2003, cited by Ke, 2008). As we can see, in this type of game, the students use mathematics throughout the game.

Another type of game that can be used with or without the help of technology are called *Jeopardy-type* games, or we could even say gameshow type games. This category implies that students simply have to answer questions to get points. Van Eck (2006) explains that this type of game is effective in promoting the learning of verbal information as concrete facts. We can therefore conclude that this type of game is a good way to check the knowledge of young people before, during or after the learning of a subject.

Methodology

Choice of participants and procedure

The methodology used for this research project is a case study. The participants for this project were 11th grade students from two math classes (of about 20 students each) and their teacher. The students as well as the teacher signed consent forms. Students under the age of 16 also needed the consent of their parents. The teacher then targeted five students of different levels of performance (to ensure a better representativeness) to participate in interviews of about ten minutes each at the end of the project. Over a period of five months (four visits), four different technology based games were integrated in the classrooms and the teacher was interviewed after every visit to collect more information. During these visits, all of the students were observed, filmed and observations were noted in the researcher's journal. The researcher also had a checklist containing motivation indicators to report how many of the five selected students (those selected for the interviews) presented each of these indicators during the integration of each game.

Choice of games

The first game introduced was the *Mathematical Battleship* which mainly follows the same rules as the basic *Battleship* game. As part of this game, students use the *Desmos* application choose coordinates to place their fictitious boats and use functions that they can enter in the application to hit the boats of their opponent. The objective of the game is to sink all the boats of your opponent before the latter can sink all your boats. This game allows students to understand the role of the different parameters of a function. For example, for a quadratic function of the form $y = ax^2 + bx + c$, if the value of the parameter a increases, the parabola becomes narrower. This game requires no preparation on the part of the teacher, but is still in the category of pedagogical games designed by educators, since it was created by me and my colleague. Below (figure 1), we can see an example of two students playing the game on their laptop.

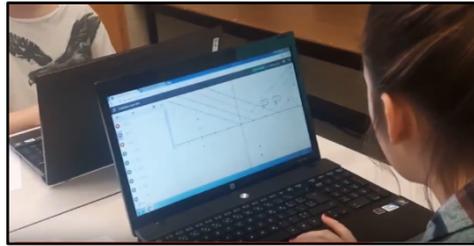


Figure 1: Students playing Mathematical Battleship

Another game that was introduced was *The Dancer and Choreographer* which is played with the help of motion detectors called calculator-based rangers (CBR). The CBR, which is connected to a graphing calculator, detects the movement of a person and the calculator represents that movement in the form of a graph. Before playing the game, students learn how to use the CBR. The activity is divided into two parts: 1) using the function of the graphing calculator called *distance match* to obtain a random graph and then 2) guiding your partner to reproduce the graph (distance in relation to time) by moving. This game allows students to understand the relationship between movement and its graphic representation. The team that faithfully reproduces the most graphs wins. This activity is a variation of the idea of Mokros (1985, cited by Lapp, 1999) who suggests that the teacher should create graphics on paper so that the students can guide their partner to reproduce them with the help of the CBR. Bellow (figure 2), we can see a student holding a graphing calculator and guiding her partner who is holding a CBR.



Figure 2: Students using the CBR and the graphing calculator.

The last two games introduced were *Jeopardy-type* games that are also pedagogical games created by educators since they were created by me. The first game is called '*Who wants to be a mathematician?*' and is created using the *Poll Everywhere* application (www.polleverywhere.com). *Poll Everywhere* allows people to create questions (multiple choice or open-ended) that can then be answered by the public by using their cell phones. Inspired by the "Ask the Audience" option in the game '*Who Wants to Be a Millionaire?*', this game allows students to answer multiple choice questions (about quadratic equations in this

case) anonymously by texting their answers which can then be projected as a graph. This game is in the student versus teacher format and to get a point, 75% of the students must have the correct answer, otherwise the teacher gets the point. The second game is *Mathematical Jeopardy* where students are in teams and have to answer questions right (about matrixes in this case) to get points, or else the other teams have a chance to get the points. At the end, the students get to bid some of their “cash” to try to double it with the final *Jeopardy* which allows groups that were trailing behind to have a chance to win.

Findings

The analysis process was based on a thematic analysis. The interviews were transcribed, the important bits were highlighted and then the information was placed in a table and separated into themes and sub-themes to then be presented in the form of a diagram.

Presence of motivation indicators

In relation to the different elements observed, it was evident that the young people were motivated during the integration of the games. The students and the teacher mentioned that the games were motivating, and one can notice this by the fact that the students were eager to come to the class, they were on task, they participated and they had fun, and according to Viau (2004; 2005), participation and pleasure are indicators of motivation. The teacher stated that the games were interesting because they change the routine and atmosphere of the class. One student mentioned that the games made him want to come to class, and this is important because, as Leblond explains (2012), when students are not motivated towards mathematics, they are disengaged from learning about them and avoid careers that are associated with mathematics. For example, some young people drop out of mathematics classes or decide to choose careers that are not associated with mathematics, because they are uninterested by mathematics. The fact that young people want to come to school if they play a game could change this situation.

The impact of technology

There are a lot of different types of games that can be used in the classroom by teachers and students. The games chosen for this study were technological games, so it is interesting to see that a motivational aspect that was emphasized by young people in the interviews is the use of technology as they become more engaged in their learning (Van Eck, 2015). This confirms the findings of Muir-Herzig (2004) who explains that students are more committed when they use

technology which we also noticed in different technology-rich contexts, like solving mathematical problems online (Freiman, 2009).

Games and understanding

Technology, in general and in the specific context of mathematics teaching and learning, provides a better level of understanding (Muir-Herzig 2004; Niess, 2016). In the interviews, several students mentioned that the games helped them better understand mathematical concepts. For example, in the function $y = ax^2 + bx + c$, variables a , b and c are parameters, so the game allows students to understand what happens if the value of one or more parameters is modified. In the context of the *Mathematical Battleship* game, the use of the *Desmos* application clearly helped young people understand, according to the teacher. Also, two students said that games allow the contextualization of formulas and show them how to apply them, and this is important, because it avoids having to memorize everything. Three students mentioned that understanding is a factor that influences their motivation, because when they understand, they find that mathematics is interesting. We observed this in several students in the classroom who decided to go beyond the framework of the game by exploring concepts they had not yet seen as new functions, radians, and inverse matrices, in order to move forward in the game. For example, in the *Mathematical Battleship* game, one student used the equation of a circle that he had just learned and one pupil decided to use the inequality $y < 6$ to sink all the boats at once. A group of four students even researched new functions so that they could use them in the game. As we can see, understanding can allow students to be more motivated and interested in mathematics and want to learn more which could help them in the future and encourage them to keep learning.

The teacher also recognized the value of games for students' understanding. Indeed, he said that one of the positive aspects of the game is that it helps students better understand the parameters of functions while providing meaningful context for problem-solving, and creating conditions for knowledge transfer, among other aspects of situated learning (Van Eck, 2015). However, the teacher was more sceptical about the role of games in students' understanding at the end of the experimentation. In fact, he said that a negative aspect of games is that they did not help students understand mathematical concepts, since the games were simply a way of reviewing material. Consequently, the teacher seems to have contradicted himself during the various interviews. This could be the result of the difference between the games, as the last two games were in the question-answer format (so more in the form of a revision), while the first two games explored a mathematical concept in particular. It is therefore possible that

these remarks were located in time and associated to the game or games that had just been exploited rather than the games in general.

The teachers and the students also talked about the use of games in the classroom and the academic performance of students. It seems that although games help students better understand mathematical concepts, it doesn't necessarily improve their results in mathematics. Actually, almost all students (four out of five), as well as the teacher, agreed that the games had no impact on their academic performance. This is not surprising given the fact that the students enrolled in these particular math classes generally have a certain ease in mathematics. Thus, a better understanding does not necessarily mean that the students will achieve a better academic result, especially if they already have good results. This finding needs to be investigated further, since contradictory findings regarding students' achievement have already been reported (Ferguson, 2014; Hung et al, 2014).

Challenges

The main negative aspect of the games was the fact that some of the laptops stopped working when the student were playing *Mathematical Battleship*, however, many students downloaded the *Desmos* application on their phone to keep playing which shows us that they were interested and persevered in difficult moments, and perseverance is one the motivation indicators underlined by Viau (2004). The other negative aspect was classroom management which, as stated by Pastore and Falvo (2010), is a common negative aspect when it comes to integrating games. When teachers try to introduce games in their classroom, they need to be certain to manage the group well, to make sure everyone is doing the task.

Limits and future research

There are some limitations associated with this research project. First, one of the limitations encountered is the fact that this research project is a case study, so it does not allow for generalization. From a technical standpoint, some of the laptops stopped working during the *Mathematical Battleship* game, and this was a major challenge as some students were not able to play the first game.

With qualitative research, it is difficult to measure student performance, so that motivation indicator was not explored in this project. Future research is needed to measure quantitatively the impact of motivation on student performance.

Conclusion

The results of this research show that games have an impact on the motivation of students. This assertion is based on the fact that students are engaged, they stay on task longer, they have fun, they are eager to come to class, they persevere when they encounter difficulties (Rosenheck et al, 2017), they are interested in having success (Medina, 2005) and they want to go further than the course framework while improving self-efficacy (Hung et al, 2014). In addition, games seem to improve students' motivation in learning mathematics. Technology also seemed to have an impact not only on motivation, but also on students' mathematical understanding (Calder, 2011) and. For example, the use of the *Desmos* application and the CBR allows students to have a visual representation of functions. There may not be a direct impact on students' academic performance, but they may better understand how certain elements associated with the subject matter (such as understanding how to apply certain formulas and how they can be represented in a different context).

Despite all the positive effects associated with playing games in the classroom, there are still some negative aspects associated with the game. Some classroom games require more preparation time and more classroom management. It is also difficult to find games that relate directly to the content being studied and that respect the learning outcomes. On the other hand, it is easy to modify already created games and adapt them to different courses (like *Jeopardy*, which is easily found online). Moreover, the creation of one game per year and the sharing of games between teachers could easily allow teachers to have a variety of games to integrate in their classrooms.

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REVISITING HARDY'S "APOLOGY": AN INTERDISCIPLINARY RENDEZVOUS BETWEEN MATHEMATICS, LITERATURE AND LITERACY

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Abstract

Mathematics often plays a role of a supporting discipline in interdisciplinary collaborations with science and social science. The question is if collaborations with the Arts may offer other roles or have the potentials to unfold other aspects of the discipline? This paper takes G. H. Hardy's aesthetic arguments for the value of pure mathematics (Hardy, 1940) as the textual basis of an interdisciplinary collaboration between the subjects of Mathematics and Danish (i.e. Language 1). Through the perspective of literacy, it points out didactic potentials in an interdisciplinary approach to beauty and aesthetics within the context of mid-20th century ways of thinking and understanding mathematics and literature.

Introduction

In many interdisciplinary educational settings, mathematics often comes to play the role of a supporting discipline in interdisciplinary collaborations with other school subjects (e.g. Jankvist, Nielsen & Michelsen, 2013; Jensen, Niss & Jankvist, 2016). Not least does this occur when mathematics is to collaborate with disciplines (subjects) within the natural sciences, since these often rely on mathematical tools in the form of applying mathematical theories, methods, etc. To some extent this goes for the social sciences as well, not least economics. From this perspective we argue that the Arts may possess more promising potentials in terms of fostering 'true interdisciplinarity'. In this paper we take the mathematician G. H. Hardy's famous essay *A Mathematician's Apology* (1940) as the outset for an interdisciplinary collaboration between Mathematics and Danish as Language 1 (L1) in the context of Danish upper secondary school. According to its regulatory documents the aim of the Danish upper secondary school (stx) is that the students should acquire "bildung, knowledge and competences through the education's combination of disciplinary width and depth and through the interplay between the disciplines" (Danish Ministry of

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Education, 2013, § 1 subsection 2, our highlighting and translation from Danish). However, the interplay between the subjects of Mathematics and Danish do not form the most frequent constellation to reach the overriding curricular goals. In his *Apology*, Hardy points at similarities and differences between aesthetics in mathematics and literature. From within the perspective of literacy, this paper will follow the potentials in Hardy's comparison, and discuss how ways of thinking and defending these two subjects (disciplines) by reference to beauty, seriousness and the significance of ideas can form the point of departure for an interdisciplinary rendezvous between Mathematics and Danish. The historical context of Hardy's *Apology* as an essay written after World War I and just before World War II of course plays a crucial role as it makes up a meeting point of mid-20th century mathematics and literary criticism. However, turning the focus to the beauty of mathematical theorems and proofs and to the ideas of modern poetry, the paper intentionally does not address aesthetics through a thematic interdisciplinarity, but rather how to give students an opportunity to understand disciplinary ways of thinking from positions inside and outside the two subjects. In this way the interdisciplinary collaboration on Hardy's essay aims at providing the students with the opportunity to see and understand beauty and aesthetics within and between the thinking of different subjects, to understand disciplinary problématiques (in a historical context), and thus to widen the school-based literacy traditionally connected to the two subjects.

Hardy's perception of beauty in mathematics

In his *Apology*, Hardy spends quite a lot of space explaining the role of mathematical beauty in the work of the pure mathematician:

The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics. (Hardy, 1940/2005, p. 14)

Hardy addresses both the beauty of a mathematical theorem and of a mathematical proof. In relation to the former, he states:

The beauty of a mathematical theorem depends a great deal on its seriousness, as even in poetry the beauty of a line may depend to some extent on the significance of the ideas which it contains. (ibid., p. 17)

As an example of one of "the most beautiful theorems of the theory of numbers" Hardy mentions "Fermat's two square theorem on the law of quadratic reciprocity" (ibid., pp. 17-18). However, since he deems the proof of this theorem

too difficult an example for non-mathematicians to follow (e.g. Euler's proof draws on quite a bit of elemental number theory), he instead turns to Euclid's theorem and proof for the existence of infinitely many primes, and Pythagoras' theorem and proof for the squareroot of 2 being an irrational number. To illustrate Hardy's perception of mathematical beauty, we provide his paraphrasing of the former of the two.

The prime numbers or primes are the numbers

(A) 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, ...

which cannot be resolved into smaller factors. [...] The primes are the material out of which all numbers are built up by multiplication: thus $666 = 2 \cdot 3 \cdot 3 \cdot 37$. Every number which is not prime itself is divisible by at least one prime (usually, of course, by several). We have to prove that there are infinitely many primes, i.e. that the series (A) never comes to an end.

Let us suppose that it does, and that

2, 3, 5, ..., P

is the complete series (so that P is the largest prime); and let us, on this hypothesis, consider the number Q defined by the formula

$$Q = (2 \cdot 3 \cdot 5 \cdot \dots \cdot P) + 1.$$

It is plain that Q is not divisible by any of 2, 3, 5, ..., P ; for it leaves the remainder 1 when divided by any one of these numbers. But, if not itself prime, it is divisible by some prime, and therefore there is a prime (which may be Q itself) greater than any of them. This contradicts our hypothesis, that there is no prime greater than P ; and therefore this hypothesis is false. The proof is by *reductio ad absurdum*, and *reductio ad absurdum*, which Euclid loved so much, is one of a mathematician's finest weapons. (ibid., pp. 18-19)

According to Hardy, a "mathematical proof should resemble a simple and clear-cut constellation, not a scattered cluster in the Milky Way" (ibid., p. 29). Euclid's proof surely fulfills this criterion; it is a proof by contradiction, and the contradiction is established by means of construction, i.e. of a prime number that is larger than the assumed largest prime number. Hardy remarks:

I said that a mathematician was a maker of patterns of ideas, and that beauty and seriousness were the criteria by which his patterns should be judged. I can hardly

believe that anyone who has understood the two theorems will dispute that they pass these tests. (ibid, p. 21)

In regard to seriousness, Hardy's argument is that primes are the very "raw material" of which we must build arithmetic, and Euclid's proof ensures us that we will never run out of "raw material".

Beauty in modernist poetry

In his defense of pure mathematics Hardy uses art and poetry (as it is evident from the quotations above) to point out and illustrate the specific character of mathematical beauty. Both artists and mathematicians make patterns of harmony and beauty, but the beauty of the mathematical patterns have a certain strength and validity, since they are made of ideas and not of words, he states (Hardy 1940/2005, p. 14). However, Hardy's comparison between mathematical theorems and proofs and the aesthetic form of poetry (and literature) is not unique. Several poets and literary critics have compared the poetic style, the accuracy of the metaphor and the abstraction of a work of art with mathematical precision and "the severe logic" in a mathematical problem (Friedrich 1956/1987, p. 35).

From our perspective of an interdisciplinary collaboration in upper secondary school education it is interesting to study how this comparison between the beauty of mathematics and that of poetry can shed light on the similarities and differences between the aesthetics of the two subjects and to pursue the didactic potentialities of the problem area. In the following paragraphs we will see the question of similarities and differences from the perspective of literature studies. The context of our discussion will be mid-20th century literary criticism, or in other words perceptions and thinking of poetry contemporary with Hardy's *Apology*.

In his classic, Hugo Friedrich (1956) highlights two tendencies in modern poetry; one which takes a free and a-logical form and another one intellectual, abstract and severe introduced by the French symbolist poet Stephane Mallarmé. Friedrich's characteristic of Mallarmé's poems has many similarities with Hardy's description of mathematical beauty. Friedrich stresses the formal precision, the sharpness and abstraction of the poems and points at a dehumanization which marks a break with a romantic poetic tradition and results in the dissolution of a recognizable reality. A clear-cut aesthetic form is very central to the poetic thinking of Mallarmé as for his inheritors in the 20th century poetry. "The one who thinks has simple hands", Mallarmé has written once (Friedrich, 1987, p. 115).

The literary critics René Wellek and Austin Warren have the same point although they find it more or less characteristic for all literature:

[...] every work of art imposes an order, an organization, a unity on its materials. This unity sometimes seems very loose, as in many sketches or adventure stories; but it increases to the complex, close-knit organization of certain poems, in which it may be almost impossible to change a word or the position of a word without impairing its total effect. (Wellek & Warren, 1956, p. 13)

But to point at the severity or seriousness of the form in (parts of) modern poetry without turning the gaze to the ideas or thoughts within this form is not to do poetry full justice. A central idea among literary critics contemporary with Hardy was the unity of form and content. Cleanth Brooks calls literature "the language of paradox" and stresses that simple literary forms like metaphors, symbols and paradoxes holds a complexity of thoughts, (not unlike Hardy's point concerning the proof of Euclid):

I have said that even the apparently simple and straightforward poet is forced into paradoxes by the nature of his instrument. Seeing this, we should not be surprised to find poets who consciously employ it to gain a compression and precision otherwise unobtainable. (Brooks, 1947, p. 10)

As the critics stress, the beauty of poetry (and literature) is not only a formal question, but also a question of how the text deals with a complexity of ideas which lie beyond everyday language, logics and reality. However in his reflection on the difficulty of modern poetry Brooks does not blame the poet but the reader's ability to read these kind of texts:

"... a great deal of modern poetry is difficult for the reader simply because so few people, relatively speaking, are accustomed to reading poetry as poetry" (ibid., p. 76)

A challenge literary pedagogy has tried to meet ever since by reflections on how to teach the reading and understanding of modern poetry and literature.

Hardy's perception of beauty in literature

As we have seen, Hardy refers to harmonic patterns in literature to highlight the nature of mathematical beauty, and literary critics refer to mathematics to describe the clear-cut form of modern poetry (and to some extent to literature in general). However, what occupies Hardy the most are the *different* ways in which patterns of ideas shape mathematical theorems and literary texts. He quotes two

verses from Shakespeare's *The Life and Death of Richard the Second* to demonstrate the differences:

“Not all the water in the rough rude sea/Can wash the balm off from an anointed King” (*The Life and Death of Richard the Second*, Act 3, scene 2)

and makes the following comments:

Could lines be better, and could ideas be at once more trite and more false? The poverty of the ideas seems hardly to affect the beauty of the verbal pattern. (Hardy, 1940/2005, p. 14)

In reading literature the perception and saying of the words may overshadow the weakness of the ideas of the verses. Hardy underlines in this way what he sees as one of the differences between clear-cut patterns in mathematics and in literature. This does not mean that ideas do not matter in poetry, but it shows, according to Hardy, that mathematical ideas hold a different seriousness and are less affected by time and history than ideas in words. He uses a quotation from another Shakespearean drama to illustrate this point:

“After life's fitful fever he sleeps well” (Macbeth 3rd act, scene 2)

Hardy finds these verses more beautiful than the previous ones because:

The pattern is just as fine, and in this case the ideas have significance and the thesis is sound, so that our emotions are stirred much more deeply. The ideas do matter to the pattern, even in poetry, and much more, naturally, in mathematics. (Hardy, 1940/2005, p. 17)

Looking at the first quotation with the eyes of the literary critics quoted in the previous section we must disagree in Hardy's judgement. When the verses from *The Life and Death of Richard the Second* are judged by their use of “the language of paradox” and of “an indirect message” (Brooks, 1947), they form a beautiful unity of a complex or ambiguous statement and a simple form. We have two points here. Firstly we agree of course that ideas in mathematics and in poetry (and literature), in spite of some similarities, take very different forms. Secondly, we think that ideas do matter to patterns in poetry as in mathematics, but to capture the similarities and differences of the beauty of the two phenomena demand different modes of thinking and reading. To study the beauties of the verses of a Shakespearean drama or the beauty of modern poetry, one needs to read “poetry as poetry” (Brooks, 1947) as well as one needs to read (and think) mathematics as mathematics in order to understand the theorems and proofs of Euclid, Pythagoras and Fermat.

In that way, an interdisciplinary rendezvous between mathematics and literature in an upper secondary school setting holds potentials to make students reflect on disciplinary ways of thinking and reading and on their common and different approaches to beauty and aesthetics as phenomena. Nevertheless, to see and conduct an interdisciplinary rendezvous in this way needs a further theoretical framing which is presented in the following sections.

Approaching aesthetics through disciplinary literacies

In this section, we argue for disciplinary literacy as a theoretical framework for interdisciplinary collaboration on beauty and aesthetics between Mathematics and Language 1 (Danish). However, before embarking on our theoretical argument, we return shortly to our introductory remarks in order to make two clarifying points about the potentials of interdisciplinary collaboration between mathematics and the Arts for moving mathematics away from the auxiliary function of a mere tool, which is the function usually conferred on mathematics when included in interdisciplinary settings.

Firstly, it is important to point out that the interdisciplinary collaboration we would suggest between Mathematics and Danish is not just about reversing the described interdisciplinary hierarchy placing Mathematics as the main subject and Danish in the auxiliary position. On the contrary, the collaboration we envisage is of a more balanced and integrative kind where both subjects will profit equally. Thus, according to Erich Jantsch's (1972) taxonomy of interdisciplinarity, the collaboration we propose would fit the characteristics of *interdisciplinarity* (proper), where the collaborating disciplines are subject to a common problem, and where the focus is on the connections (similarities and differences, e.g. the preceding section) between the collaborating disciplines (Jankvist, 2011).

The second point is related to the first. A parallel can be discerned between Mathematics and Danish regarding the pedagogical challenge related to aesthetics and beauty as an essential dimension of the two disciplines. In the introductory paragraph we described this challenge in Mathematics, so here we turn briefly to Danish. Aesthetics is a core element of the curriculum for Danish in upper secondary school, and it figures as an essential characteristic in the general description of the discipline (e.g. Ministry of Education, Regulatory document for upper secondary school, Danish, §1 subsection 1). However, at the same time, there is ample documentation to fact that the aesthetic dimension does not easily come across to the students, especially with regard to literature and (modernist) poetry. It is quite commonplace to see quotes by students stating that their appreciation of literature has been killed rather than enhanced by the teaching of Danish, and that they cannot see the point in dwelling on the the

enigmatic wordings in poems by for example Arthur Rimbaud: “Why can’t he just say it as it is!”, they object. Thus, the aesthetic dimension is a pedagogical challenge in both Mathematics and Danish, and though this challenge is different in nature – in Mathematics, the aesthetic dimension is often neglected; in Danish it does not easily transmit to the students – it constitutes the basis for the balanced collaboration between the two disciplines.

We now turn to the discussion on disciplinary literacy as a theoretical framework for the interdisciplinary approach to beauty and aesthetics. While acknowledging that the concept of literacy has been and still is the subject of much academic debate and dispute as to its proper meaning and use, we shall refrain from participating in this intricate theoretical discussion in this paper. When using the term literacy, we will simply refer to the definition proposed by Dagrún Skjelbred and Aslaug Veum: “Literacy signifies the ability to interpret, produce and reflect on texts” (Skjelbred & Veum, 2013, s. 18, our translation). Following from this definition, disciplinary literacy designates the specialized literacies connected to different content areas, professions or domains, and it is characterized by distinct discursive conventions in the form of specialized ways of using the verbal language as well as other semiotic modalities to shape knowledge and meaning (Bremholm, 2016; Jetton & Alexander, 2004; Shanahan & Shanahan, 2008). Several studies in disciplinary literacy point to this intrinsic interdependence between disciplinary content and discursive form as a particular demanding challenge, when students, or people in general, learn and try to master a specific subject or discipline:

The complexity of the process of learning to be literate in a content area lies in the fact that these skills are interdependent. That is, being able to access content knowledge depends at some level on one’s understanding of the discursive conventions of the content area. In a similar manner, developing strong interpretive or rhetorical skill in a content area requires that one understands the relevant content concepts. (Moje et al., 2004, p. 45)

Interdisciplinary potential in a literacy perspective

If now, we contemplate the question of mathematical and poetic beauty and aesthetics from the perspective of disciplinary literacy, the potential of an interdisciplinary approach becomes apparent. This is the case because the aesthetic dimension of mathematics and literature explicitly illustrates how discursive form and subject related content are interwoven in specific ways in the two subjects, and thus it constitutes a direct and easy accessible manifestation of the disciplinary literacy of the two subjects. In this perspective, Hardy’s *Apology* represents an exemplary text for at least two reasons. Firstly, because we

witness, in this text, a mathematician's explicit reflections on the kind of relationship between content and form that represents the specific beauty of pure mathematics, and in this sense you might even say that Hardy's *Apology* constitutes a metatext on disciplinary literacy (of mathematics). Secondly, because Hardy's essay explicitly invites the interdisciplinary perspective by comparing mathematical and poetic beauty.

In regard to the first reason, we saw earlier, using Euclid's proof as our example, how the relationship in question is characterized by "a pattern of ideas" fitting "together in a harmonious way" or in "a simple and clear-cut constellation" thereby showing us pure mathematical reality. It is interesting to note that Hardy, when speaking about the beauty of mathematics, uses the term "ideas" and not "form" or "language", which indicates that he adheres to an essentialistic notion of mathematics that does not recognize the discursive, or rhetorical or – as we would say in this paper – the literacy aspect of mathematics. However, his metaphors give him away, so to speak. When Hardy describes the arrangement of ideas in Fermat's, Euclid's and Pythagoras' theorems and proofs as patterns and constellations, it is a roundabout way of admitting the importance of the semiotic or discursive arrangement of these ideas which, following the explanation above, points to an aspect of the specific disciplinary literacy of mathematics. Furthermore, as the explanation of Euclid's theorem presented earlier exemplifies quite clearly, mathematical thinking (constituting patterns and constellations) manifests itself amongst other by the use of a specialized language and mathematical notation, i.e. disciplinary literacy.

As to the second reason, we saw in the paragraphs about the aesthetics of modernist poetry that even though Hardy's comparative approach to mathematics and poetry helps bringing to the fore the aesthetic dimension of mathematics, it also represents a simplistic view of poetic beauty, as understood by Hardy's modernist contemporaries, that is not really sensitive to the important differences that also exist between mathematics and poetry. So, within the perspective of disciplinary literacy, Hardy's *Apology* is interesting on the one hand because its comparative approach helps elucidate what constitutes mathematical understanding and discourse of beauty, and on the other because it shows us the difficulty, or maybe even impossibility, of a given disciplinary literacy (in this case mathematics) to represent and grasp a phenomenon outside its disciplinary boundaries (in this case poetry).

In an educational setting in upper secondary school, the described qualities of Hardy's *Apology* could advantageously form the textual basis for the interdisciplinary collaboration between mathematics and Danish (in the sense of interdisciplinarity proper). Thus, having students crossread Hardy's text – preferably in combination with chosen extracts about modernist literary

aesthetics, e.g. from Friedrich or, perhaps even better, from poetics by modernist poets (e.g. Baudelaire, Eliot or Sterne) – in Mathematics and Danish would provide the students with a rare opportunity to reflect upon and discuss aesthetics – both as a general phenomenon (the interconnectedness of form and content), and, more importantly, as a specific aspect of the disciplinary literacy of the two subjects.

Through the interdisciplinary collaboration with its double perspective the aesthetic literacy of the two subjects might mirror each other, and thereby permit the students to apprehend the specific nature of each by letting them reflect upon what is common and what is different vis-a-vis the other.

Thus, when the students contemplate mathematics mirrored by literary or poetic literacy it might help them to recognize and better understand what aesthetics is in relation to mathematics, and how it constitutes an essential dimension of (pure) mathematics (i.e. the necessary and harmonious interconnection between (mathematical) content and form (semiotic or discursive representation), where this form provides a unique and perfect shape to a content that exist as mathematical reality prior to its representation). Similarly, looking at modernist poetry and poetics reflected by Hardy's mathematical aesthetics might function as a didactic opening that supports the students' appreciation and understanding of the poetic language and aesthetics as an intrinsic part of literature itself. In other words, discussing and comparing with the mathematical examples of the necessary interconnection between mathematical content and discursive form provided by Hardy, might help the students' realize that the modernist poet cannot "just say it as it is", because if he tried "it" would no longer be the same. Furthermore, the interdisciplinary comparison and reflection might at the same time lead the students to distinguish between mathematical and modernist poetic aesthetics, and to understand that the latter in contrast to the former is characterized by a complex and paradoxical parity between content and form, where the content does not exist prior to the discursive form. Instead the poem constitute a new reality inasmuch as its unique unity of content and form brings forth an aspect of reality that did not exist before the poet created the poem by putting the words on paper.

Concluding reflections

It should be mentioned that in this paper we have not touched much upon the historical context of Hardy's essay, although this clearly would need to be done in a given educational setting and implementation. Likewise, we have knowingly eschewed the didactic question of how to practically organize the interdisciplinary collaboration, as well as the problem that the traditional school-based literacies of the two school subjects might pose a resistance to this kind of

interdisciplinary approach. Our purpose with the present paper has been to address the potential educational benefits of establishing a rendezvous between the disciplines of Mathematics and Danish (literature) from a literacy perspective. Hence, to wrap up our line of argument, we have attempted to show how such an interdisciplinary collaboration centered round Hardy's *Apology* might foster a didactic situation, where the two subjects mutually reflect/mirror each other, and thus provide an opportunity for the students to contemplate and reach an understanding of mathematical and modernist aesthetics. Furthermore, we have argued that the students through this contemplation may come to obtain a deeper insight into the thinking (and epistemology) of the two subjects, as well as into the close interconnection between this thinking and the specific discursive form of the subjects. Thus, we claim that the interdisciplinary collaboration permits the students to access and reflect upon the disciplinary literacy of mathematics and poetry/literature. In this sense, and with reference to James Paul Gee's (2008) notion of (disciplinary) literacy as being/doing a specific social identity, we suggest that the interdisciplinary collaboration on Hardy's *Apology* has the potential of letting the students apprehend what being (doing) a mathematician and a poet actually means, or in other words, what it means "to read mathematics as mathematics" and "to read poetry as poetry".

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AN INVITATION TO MATHEMATICAL MODELLING OF ARTISTIC SPACE IN LITERARY CRITICISM: MASOCHISM RECONSIDERED

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Abstract

Mathematical methods have been employed in literary criticism for more than a century. The vast majority of the works using mathematical approaches in literary criticism is based on statistics. However, statistical methods are not the only mathematical approaches producing new results valuable for literary criticism. Literary criticism often use mathematical and physical terms and objects metaphorically. Such is 'artistic space', a term attracting great attention. Various theories of space in narrative texts were developed by many scholars, including most notably M. Bakhtin, Yu. Lotman, and G. Zoran. Their use of 'space', which is a fundamental concept in hard and life sciences, motivates one to enrich our understanding of special phenomenology by using mathematical tools. Taking as an example two world-famous masochistic novellas, we demonstrate how mathematical modelling reveals the structure of the artistic space in such kind of narrative. The modelling of love dynamics via differential equations, as well as the application of cluster analysis, reveals the undoubted similarity of 'Venus in Furs' by Leopold von Sacher-Masoch and 'Torrent of Springs' by Ivan Turgenev. Our Analysis shows that the artistic space of the masochistic text is divided into two sub-spaces, each having its own peculiarities. Though a mathematical method cannot be viewed as a 'rigorous proof' in literary theory, the suggested interdisciplinary approach allows one to compare the two love plots in such a way as to highlight their intrinsic topical, and not just stylistic, likeness as well as typological affinity.

Introduction

Mathematical methods have been employed in literary criticism for more than one and a half century now, their history beginning with the letter of Augustus de Morgan to his friend William Heald (1851) in which he assumed that the texts

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of different authors of the same historical period (e.g. Herodotus and Thucydides) demonstrate different average word length. Later similar ideas were presented by Thomas Mendenhall (1887) and Conrad Mascoll (1888). The demonstration of statistical method in stylometry can be found in Andrei Bely's *Symbolism* (1910) where formal statistical method was applied to the study of poetry. The approaches based on statistics still prevail in literary studies (see, for instance, the research of Stanford Literary Lab and Franco Moretti (2013)). Also, the mainstream of such researches is the authorship problem. The powerful mathematical machinery, including data mining technique, artificial neural networks, etc., is used for this purpose (Khosmood & Kurfess, 2005; Zhao & Zobel, 2007; Koppel et al, 2009).

While recognizing the progress made in this field, we ask, what other mathematical approaches can be useful in the field of literary theory? Statistical methods are not the only mathematical approach producing new results related to literary criticism. An attempt of general consideration of the relation between mathematics and literary criticism can be found in the article of Richard Schoek (1968). In particular, he was critical of metaphorical usage of mathematical terms in literary criticism. He discussed the application of some specific methods (topology, vector analysis) to literary criticism.

Mathematical modelling of literary plots can reveal new crucial aspects of a piece of fiction under study. This fact appears to be quite surprising, since mathematics, on the one hand, and literary criticism, on the other, have been historically aimed at entirely different matters. This is the underlying reason why there are so few studies in which mathematics 'serves' close reading, despite the fact that some prominent scholars urged their colleagues to use mathematical approaches in literary criticism, as Yuri Lotman did (1997).

What are the prerequisites for applying mathematical modelling in literary criticism? First of all, any literary text can be viewed as a model. The writer simulates his/her perception of the world, his aspirations, some specific situations, etc. But this kind of modelling is still quite unlike the modelling in mathematical sense. Nevertheless, we can easily find the ideas that guide us to mathematical modelling. 'Artistic space' is one of such concepts. Historically this idea is rooted in the chronotope theory of Mikhail Bakhtin (1937). Various theories of artistic space and literary space (Bakhtin, 1920-1930; Blanchot, 1955; Lotman, 1968; Uspensky, 1970; Zoran, 1984) display general features.

Artistic space differs from space in physics or mathematics. It includes dissimilar elements such as loci, characters, storylines, etc. Also, it can include remembrances, fantasies, dreams, etc. All this shows that artistic space is multi-dimensional and that its dimensions are of different origin. Nonetheless, the concept of space suggests some structure. The apparent completeness of a piece

of fiction allows one to view it as a model of the world. Indeed, the 'plotless' piece of art has certain structure and, consequently, some structured artistic space. For this reason it is tempting to use corresponding mathematical machinery to study the structure of artistic space. Graph theory, methods of algebraic topology, and some other approaches can be used for the investigation of artistic space structure. An example of skillful and interesting analysis of space and time aspects in Turgenev is found in the book by Elizabeth Allen: "Turgenev artfully manipulates the spatial and temporal dimensions of his narratives in order to set very deliberate physical, metaphysical, and moral parameters within which the events he portrays can distinctively unfold. [...] In Turgenev's fictional universe, space is not an infinite extension and time is not an endless continuum. Both are portrayed as confined and confining." (Allen, 1992, p. 73)

'Dynamics' is another concept found in literary theory. For instance, Zholkovsky and Scheglov (1987) use such terms as 'path', 'jump', 'sudden turn', 'dynamization'. Lotman speaks about 'crossing the border' (1977). Another example of the discussion of dynamics is Wolf Schmid's *Narratology* where he claims: "Literary theories must do more than just register the presence of changes of state. Even the shortest stories, [...] will represent a vast number of changes. [...] We require categories which allow us to distinguish between the countless natural, actional, and mental changes – from thunderclap to victory in battle to a hero's moral conversion or psychological transformation." (Schmid, 2005, p. 8)

We use both concepts - *dynamics and artistic space* - in our research. We distinguish three stages in our approach to modelling. First, we map an object (artistic space) into certain mathematical set (often a space in mathematical sense). Then we investigate this mathematical space by mathematical methods. The last stage is mapping the result into the initial object, translation from mathematical language into the language inherent to the object. There are no definite prescriptions for the first and the third stages of the investigation. The mapping of artistic space into a mathematical object, as well as the turning of mathematical results back into the language of literary criticism, is a kind of art. The novelty and the credibility of the results are the basis for the estimation of a particular research.

In the present paper, we demonstrate the potential of mathematical modelling in literary criticism by employing both differential equations and cluster analysis to study two novellas. The first one is *Venus in Furs* by Leopold von Sacher-Masoch (1870), the author who gave his name to the term 'masochism'. The second one is the *Torrents of Spring* (1871), the novel by Russian writer Ivan Turgenev, the favorite author of Henry James (1873; 1896) and Joseph Conrad (1921). Turgenev's collection of short stories pointed against slavery and serfdom *A Sportsman's sketches* (1852) became the Russian equivalent of *Uncle Tom's Cabin*

(Stowe, 1852) and made its author famous as a writer, public figure and human rights champion in the 19th century.

Our interest in masochism is caused by the following facts. The phenomenon of masochism is culturally widespread as it is based on the imitation of the social submissive relations that are political rather than sensual in their origin, like those between the master and the slave, the sovereign and the vassal, etc. Masochistic relationship, though belonging to a variety of romantic relationships, has its own distinctions. The chosen novellas exemplify two different masochistic experiences: the one that can be considered as purely romantic and the other based on the repetitive imitation of social relationships. We intend not only to demonstrate the typological likeness of the novellas and clarify important aspects of their structure, but also to elucidate the subtle distinction in the origin of masochistic relationships.

1. The Analysis of Love Dynamics by means of differential equations

We demonstrate the potential of mathematical modelling in literary criticism by employing differential equations to analyze two novellas: *Venus in Furs* (1870) by Leopold von Sacher-Masoch and Ivan Turgenev's *Torrents of Spring* (1871). The similarity of these two texts, as well as the resemblance of their authors' personalities, was emphasized in the book by Larissa Poluboyarinova (2006).

Mathematical modelling of love affairs dates back to the articles of mathematicians (Strogatz, 1988; Rinaldi et al., 2013; Rinaldi et al., 2015). Their aim was to illustrate the possibilities of love dynamics modelling using the systems of differential equations. We apply this approach to analyze the love dynamics in the novellas and then to connect the models with the ideas and methods of literary criticism. The model is constructed in such a way as to render major - obvious - features of the dynamics of the relations between the characters, such as the growing attachment of Dmitri Sanin to Maria Nikolaevna in Wiesbaden or *qualitative* 'love/enmity' relations between Severin and Wanda. There can hardly be offered a quantitative scale to measure feelings between either characters in fiction or between real persons. Still, we use the relations like 'love/hatred', 'greater/less' to make a judgment about love dynamics. The analysis of the differential equations solutions, which model the relationship dynamics reveals previously undetected very important features of the artistic space structure. It should be noted *that the discovered features were not embedded in the model in the process of its construction.*

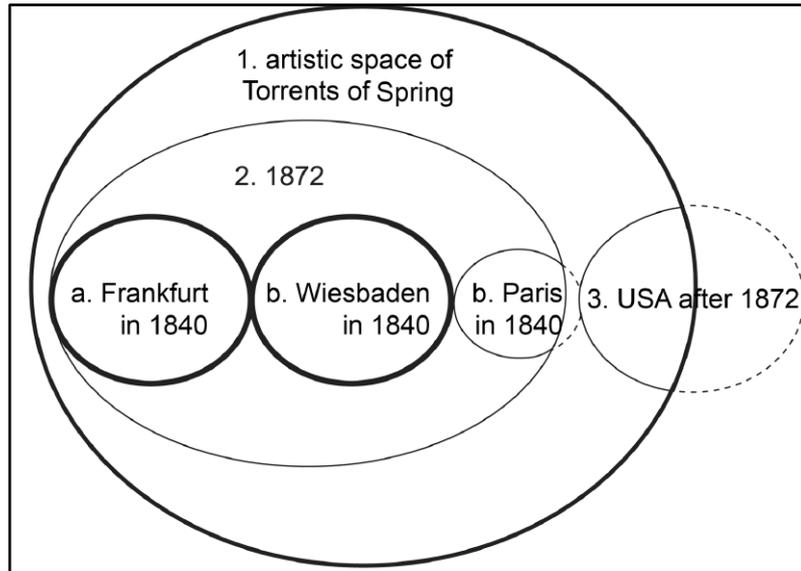


Fig. 1. Projection of the artistic space of *Torrents of Spring* onto 'loci subspace'.

We try to avoid metaphorical use of mathematical concepts. Still, sometimes what looks like a metaphor is actually an element of new construction. The following example clarifies this idea. We have emphasized that artistic space includes heterogeneous elements. For our purposes, we do not need to take into account all elements, all relationships between the elements. What we need can be called the 'projection' of the whole artistic space onto some subspace. It is convenient in our case to 'project' the artistic space onto locus subspace. The loci we distinguish in the artistic spaces of *Torrents of Spring* and *Venus in Furs* are presented in Figs 1 and 2 correspondingly. Further investigation of this 'projection' is based on the modelling of love dynamics by means of the systems of differential equations.

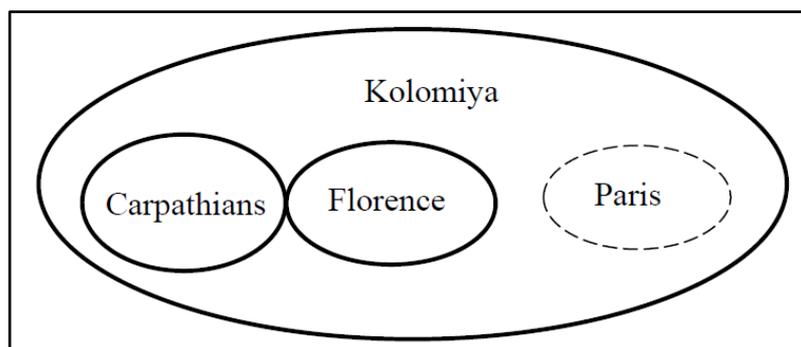


Fig. 2. Projection of the artistic space of *Venus in Furs* onto "loci subspace".

The detailed explanation of the equations' construction can be found in our work (Zhuravlev et al., 2014). The solutions of the equations give the time dependence

of feelings. These functions are presented in Figs. 3 and 4. The analysis of the obtained results proves that *Torrents of Spring* and *Venus in Furs* can be equally labeled as 'masochistic' texts; their love plots revealing the unmistakable likeness. We have summed up the typological features of the mathematical model reproducing the specific dynamics of 'masochistic' love in fiction. They are as follows: 1) the function of distance underscoring the importance of the closeness of the protagonist to the object of his perverse passion. 2) The crucial difference in love dynamics in the two subspaces of the general artistic space in masochistic novellas. The subspaces are not necessarily connected with loci. 3) The extraordinary character of relations of the masochistic couple in stationary regime is expressed in its oscillatory nature. 4) The role of masochistic pact. The above-mentioned subspaces specified in the mathematical model and the corresponding regimes of love relations are separated in *Venus in Furs* by a highly symbolic event – by signing the masochistic contract, and in *Torrents of Spring* – by the moment when Sanin accepts an iron ring from Maria Nikolaevna. The graphs presented in Figs. 3 and 4 demonstrate this similarity.

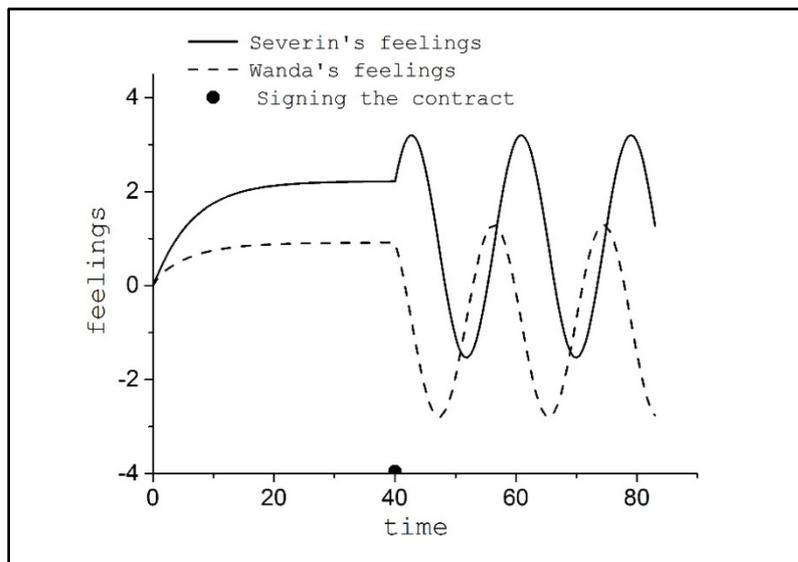


Fig. 3. Time dependence of Wanda and Severin's feelings. The borderline between the areas of different time dependence correlates with the moment of Severin's signing the masochistic contract (This figure is taken from Zhuravlev et al., 2015).

We would like to emphasize how accurately mathematical modelling, as we happened to find out, indicates the point on the temporal axis that correlates with the moment of signing the contract. Without mathematical modelling, we would not be able to indicate the moment of offering the ring in Turgenev's text as a starting point in the new modus of the relations between characters, on the one hand, and as an equivalent of signing the masochistic contract, on the other.

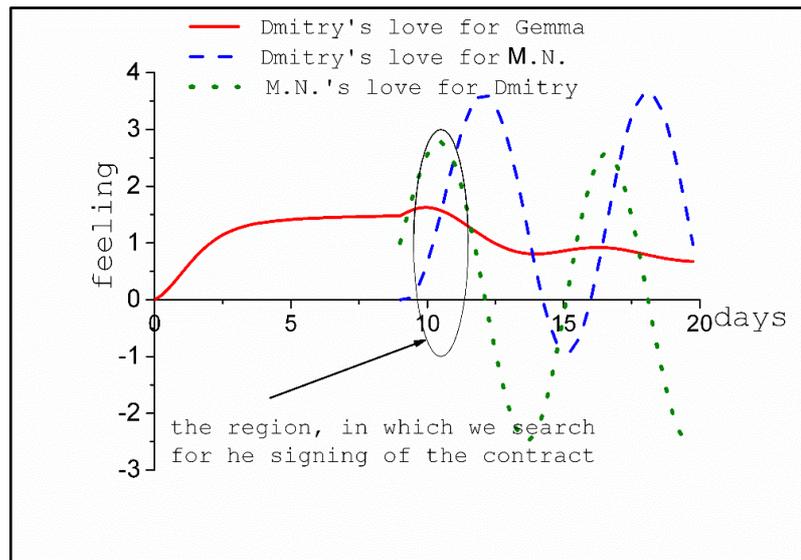


Fig. 4. Time dependence of Dmitry Sanin, Gemma and Maria Nikolaevna (M.N.) feelings in Torrents of Spring. We search for the analogue of masochistic contract signing in the vicinity of the borderline between the areas of different types of love dynamics.

Indeed, at first glance, the turning point in the love affair of Sanin and Maria Nikolaevna seems to be the love scene in the mountains. Without our mathematical model, we would wrongly treat the handing of the ring to Sanin as a form of marriage proposal. Neither could we notice that such dramatic turn in Sanin's fate does not strictly correspond with the border between the subspaces of Frankfurt and Wiesbaden. All this illustrates the difference between 'pure' close reading, still rather popular in literary criticism, and our modelling. The suggested interdisciplinary approach allows one to compare the two love plots in such a way as to highlight their intrinsic topical, and not just stylistic, likeness. This method also reveals typological affinity that we determine as specifically masochistic.

2. Q-analysis for Masochistic Novels

Among various versions of cluster analysis, the one proposed by Ron Atkin (1981) is especially suited for literary study. Atkins named his version of cluster analysis 'Q-analysis'. It has been thought that Q-analysis can be applied primarily to the investigation of urban economy, management, economy planning and other problems related to large-scale systems. However, it was Atkin, who demonstrated a possible application of Q-analysis to the investigation of literary texts. (cf. Casti, 1979) Another important work where Q-analysis was applied to study a literary text is Lawrence O'Toole paper (1980), which presents the analysis of the *Book of Genesis* and that of James Joyce's novella "Eveline" from *Dubliners* (1914). O'Toole went into details when he considered

such an important procedure of the analysis as the definition of semiotic axis in semiotic space. The term ‘semiotic space’, as it appears in O’Toole’s article, can be viewed as synonymous to ‘artistic space’.

Following O’Toole, our first step in constructing the semiotic space of a text is to establish a hierarchy inside the groups of homogeneous elements. The groups constitute the semiotic axis. Fig. 5 gives the example of such axes in the semiotic space of *Torrents of Spring*. For instance, consider the axis, which represents the Values. We distinguish two elements on higher levels: Christianity and Paganism. The next level includes such elements as Cross, Snake, Garden of Eden, Aeneas, Sorceress, etc., referred to in the text of the novel. Each element of this level belongs to either sphere of Christianity or that of Paganism. The other two axes in Fig. 5 represent loci and characters.

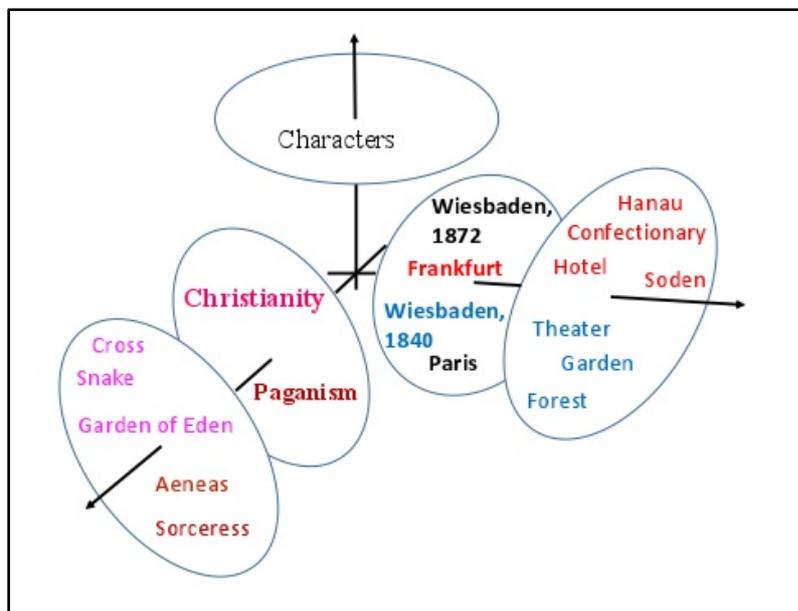


Fig. 5 Visual representation of the semiotic axes in the semiotic space of *Torrents of Spring*.

Comparing this representation with the analysis performed in the preceding section, we find that the love dynamics was considered in a specific hierarchical level of loci (cities – Frankfurt and Wiesbaden where the stages of love story differ tremendously). So, one can conclude, that the ‘city’ level of the Loci axis constitutes the inbuilt ‘natural scale’ of love dynamics in Turgenev’s novella. It is worth mentioning that O’Toole referred to the attempts of Yuri Lotman and Roland Barthes to relate spatial opposition to plot dynamics.

Now we demonstrate how the approach developed by Atkin and O’Toole can be applied to the analysis of the chosen novellas. Q-analysis represents the elements of a system and the relations between them in the form of simplicial complex. Simplicial complex is a multigraph, i.e. a set of nodes connected by the edges.

Two nodes may be connected by more than one edge. In our case, a set of nodes is a set of homogeneous elements of artistic space. An edge represents a relation between the elements of artistic space. The relations are chosen and mapped in the process of reading. For instance, the characters may be grouped according to the loci where they appear. This relation can be represented as an incidence matrix – the matrix whose elements are either 0 or 1. ‘1’ means that the character is present in this locus. The Loci and Characters-Loci incidence matrix for *Venus in Furs* is as follows:

Symbol (Fig.6)	Wanda W	Severin S	Wanda’s friend F	Painter P	Alexis A	Narrator N	Venus V
Kolomea 	0	1	0	0	0	1	1
The Carpathians 	1	1	1	0	0	0	1
Vienna 	1	1	0	0	0	0	1
Florence 	1	1	0	1	1	0	1
Dream 	0	1	0	0	0	1	1

Table 1. Incidence matrix for Characters-Loci simplicial complex for Venus in Furs.

Each locus determines a simplex (or cluster). We include not only geographical locations but also the dreams in the set of the loci. If we combine all these simplexes in one figure, we obtain a simplicial complex (Fig. 6).

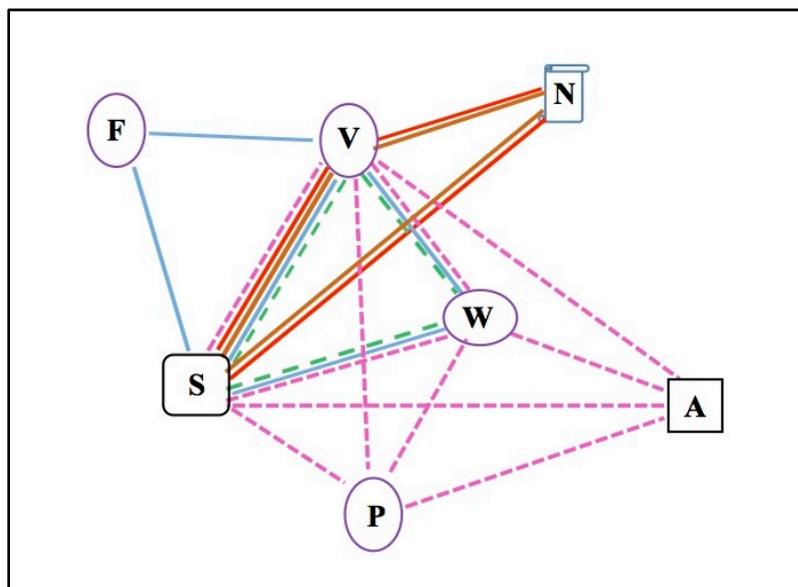


Fig. 6 Simplicial complex based on Characters-Loci relations for *Venus in Furs*.

Considering Fig. 6, we find that this simplicial complex reveals that Severin and Venus are present in every locus. We have to underscore that Venus is a metaphorical representation of Severin's erotic fantasies. She appears first as a figure in Titian's painting, then as a Greek goddess in the protagonist's night-dream. The artist who paints Wanda's portrait also indicates Venus. He refers to the legend of medieval German poet Tannhäuser³ without mentioning his name. According to this legend poet was captured by Venus, the pagan goddess who turned into the sorceress and lives in exile in a magic mountain Venusberg. Apparently Venus is not identical to Wanda in artistic space of Masoch's novella. However, Wanda appears alongside with Venus in every locus. This fact proves that Wanda always features as a hypostasis of Venus. Such pervasive accent on symbolic, mythological representation of erotic fantasy is exemplary of Masoch's method. This representation of 'masochistic' love as *idée fixe* is quite different from Turgenev's method.

The loci seem to be a most natural relation (from the mathematical point of view) connecting the characters. Nevertheless, other relations bear important information concerning the semiotic organization of the text. To illustrate this statement, let us consider the Characters-Values simplicial complex for *Torrents of Spring* (Fig. 7). The corresponding incidence matrix and the notations are

³ The story of this poet became famous after Richard Wagner's wrote his opera *Tannhäuser and the Minnesingers' Contest at the Wartburg* (1845).

presented in Table 2.

Symbol (Fig.7)	Dmitr y Sanin D	Maria Nikola -evna M	Gemm a G	Ippolit Polozov I	Dönhof D	Emilio E	Lenore L	Panta leone P	Klüber K
Love —	1	0	1	0	0	0	0	0	0
Passion —	1	1	1	0	1	0	0	0	0
Family - - -	1	0	1	1	0	1	1	1	1
Slavery —	1	1	0	1	1	0	0	1	1

Table 2. Incidence matrix for Characters-Values simplicial complex for Torrents of Spring.

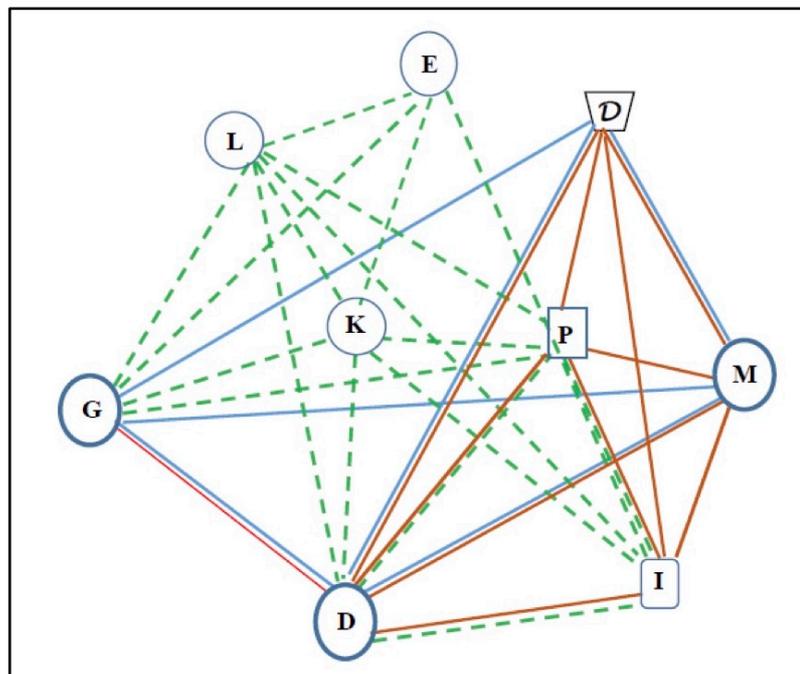


Fig. 7. Characters-Values simplicial complex for Torrents of Spring.

This simplicial complex shows strong correlation between Passion and Slavery. We believe that this correlation demonstrates how strong a masochistic vein is in Turgenev's world picture. This finding is supported by psycho-biographical study of Turgenev [Peace, 2008]. Also, this complex shows that Sanin and Panteleone are marginal in the Family pool (see the Family edge in the simplicial complex, Fig. 7). Indeed, they have no families and there is very little hope that they will.

We have analyzed only one possible simplicial complex for each of the two chosen texts. More complexes can be constructed for both novellas that would reveal the typology of dramatis personae (their beliefs, values, morals, etc.) and highlight the links between them.

Conclusion

In the present article, we have demonstrated how mathematical methods can be used to study the structure of narrative texts. We applied two different mathematical methods for the investigation of two novellas. Both of them speak about transformation of the romantic feeling. The first one, *Venus in fur* (1870) is notoriously famous for being masochistic. The second one, *Spring torrents* (1871) describes the similar romantic situation and supposed to be masochistic too.

The essential element of our research was the artistic space – the concept, introduced and analyzed in several theoretical works. We demonstrated how useful this concept is for the application of mathematical modelling in the research of various layers of meaning in the works of literature.

The analysis of love dynamics by means of the systems of differential equations has allowed us to objectively compare two love stories in such a way as to show their essential, and not just stylistic similarity and the analogous mode. It means that we prove that *Spring torrents* is also a masochistic novel. Rather, mathematics reveals a typological kinship in the construction of the masochistic topos. We have demonstrated that our modeling has prognostic potential. Differential equations and the graphs constructed on their basis are not a 'mathematical paraphrase' of the plot. They have a general meaning, showing the essence of the model of the artistic world in a particular work. In principle, this kind of reading is able to predict the future of the characters, the mode of their potential destiny. We also draw attention to the fact that the use of mathematical models creates a convenient framework for comparing texts.

Considering the dynamics in romantic plots, we not only clarify or literalize the concept of 'artistic space-time' or graphically represent the evolution of characters. Rather, the above mathematical interpretation of the plot (we

emphasize that this is not a quantitative but a qualitative method) is an alternative way to represent meanings and composition, if not authorial intentions, such method being inherently analogous to more traditional reading modes. Still, the proposed method brings about new knowledge.

Using the two masochistic novellas as an example, we showed the Q-analysis machinery is a promising instrument for the investigation of artistic space. The analysis of the simplicial complexes based on the constructed axes of semiotic space allows one to draw solid conclusions concerning the structure of the artistic space of the text. Moreover, simplicial complexes clearly demonstrate the focal messages that are to be found in the narrative structure. Finally, this method has considerable advantages allowing the critic to make an objective comparative research of the texts without referring to intertextuality. In our opinion, Q-analysis is quite universal comparing with other mathematical methods applied to literary studies. In general, mathematical modelling seems to have wide potential application in literary theory. It can be based on various branches of mathematics. As a consequence, not only the statements obtained in the framework of traditional literary studies can be verified but also new results can be obtained.

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RELATING MATHEMATICS AND LITERATURE AS A TEACHING STRATEGY AT THE HIGH-SCHOOL LEVEL

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Aguilar¹

Abstract

We report the application of a didactic design that relates literature and mathematics and is embedded in the paradigm called «questioning the world» as proposed by Chevallard (2013). The design was implemented in two groups from a high school located in a marginalized urban area of Mexico. The heart of this activity is the analysis, from a literary as well as from a mathematical point of view, of the story of Jorge Luis Borges “The library of Babel”. Some of the elements analysed by the students were the form of the Library of Babel, and the number of galleries, books and pages contained in it, which in turn motivated the articulation of ideas about the notion of infinity. Besides pretending that students explore the mathematical ideas involved in the story, the aim of the design is also to promote a critical thinking among students and provide them with tools to question their reality.

Introduction

The use of literature into the teaching of mathematics is not a new concept, different scholars have argued about the benefits of integrating these two disciplines. For example, through the use of literature in the mathematics classroom students’ interest in literature and mathematics can be fostered, but such integration also facilitates the introduction of new mathematical ideas in the classroom, it encourages the development of critical thinking, and promotes the understanding of new mathematical concepts. In this manuscript we report a teaching experience at the high school level that is based on the integration of literature in the teaching of mathematics.

The teaching experience consisted on the application of a didactic design called “A library for your community” that had a dual purpose: on the one hand, to

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engage students in the analysis of a literary work that contains mathematical notions, and to promote the understanding of such notions; on the other hand, to foster students' reflections on the characteristics and role of libraries in their community.

To place our work in the landscape of research on the use of literature in the teaching of mathematics, we present a brief review of this area of research. Later we clarify the theoretical position in which our experience is framed. We then detail the method followed for the implementation of the teaching experience. Finally we show some of the results of the implementation, and we mention some conclusions.

Use of literature in the teaching of mathematics: a brief review

For at least 25 years, there has been an interest on the part of mathematical educators from different regions of the world to use literature as a means for the teaching of mathematics (see for example Welchman-Tischler, 1992). Although most research on the use of literature in mathematics teaching has been developed at the preschool and elementary school levels (see Flevaris & Schiff, 2014, for a comprehensive review on teaching and learning early childhood mathematics with children's literature), there are some reports of teaching experiments at the high school level (for example Sriraman, 2003; 2004).

Researchers have reported different benefits of integrating literature into mathematics teaching. Such benefits can be classified into two categories: *learning benefits* and *affective benefits*.

Learning benefits of using literature to teach mathematics

One of the benefits that have been identified is that the use of literature in the teaching of mathematics can promote students' problem solving and problem posing abilities. For instance, Young & Marroquin (2006) illustrate how mathematics teachers can pose rich mathematics problems through their contextualization in scenarios and stories from children's literature. These researchers conclude that such literature-based problems not only enhance problem-solving skills, but also encourage communication and justification processes.

Another benefit reported in the literature is the development of students' critical thinking. An example of this is the work by Sriraman (2013), who asserts that critical thinking is a common element between literature and mathematics. This author organized a teaching experiment of several sessions in which 13-14 year-old students read and analysed the contents of Edwin Abbott's Flatland book. Sriraman (2013) argues that this activity allowed students to develop their critical

thinking, and to be introduced to sophisticated mathematical ideas such as non-Euclidean geometries.

It has also been suggested that books have the potential to encourage students to make connections between mathematical ideas and their experiences outside of school. For example, Hellwig et al. (2000) illustrate how trade books (this is, books other than textbooks, in mathematics instruction) can help students to connect mathematics with other content areas, and to identify connections between different ideas in mathematics.

Affective benefits of using literature to teach mathematics

Another merit of the use of literature as a means to teach mathematics is that it can produce emotional and motivational gains in students. An example of this is reported in the study by van den Heuvel-Panhuizen & van den Boogaard (2008) where they read picture books to young children to stimulate mathematical thinking about geometry, data representation and measurement; however, these researchers claim that these books also have the power to engage and focus the attention of a group of children, which in turn can facilitate interactions between the children. Similarly, Hong (1996) analyse the effectiveness of using children's literature to promote mathematics learning, particularly the effects on dispositions toward mathematics learning, concluding, "the disposition to voluntarily pursue mathematics learning can be increased using children's literature" (p. 488).

The use of literature in the teaching of mathematics also has the potential to support the mathematical learning of students with different backgrounds and abilities. Iliev & D'Angelo (2014) assert that the inclusion of multicultural literature in mathematics classroom "enables children to move beyond their current knowledge base and make culturally relevant mathematical connections through such universal topics as money, time, and measurement" (p. 454). In turn, Courtade et al. (2013) claim that shared story reading, defined as: "when a teacher orally reads a book, and students purposefully and strategically interact with both the content of the book and the teacher" (p. 35), can be an effective mean to support the mathematical learning of students with moderate and severe disabilities. Moreover, Casey et al. (2008) suggest, "integrating literacy and mathematics through oral storytelling can enable low-income, diverse learners to acquire better geometry skills" (p. 43).

The aim of the teacher experience that we report in this paper was to generate both, learning and affective benefits for the students. On the one hand it was tried that the students analysed and deepened in mathematical notions such as infinity and the properties of some geometric forms; but on the other hand, we also

sought to promote students' critical thinking, particularly in connection to the role that libraries play in their own community. Finally, we also wanted to foster a perspective among the students where literature and mathematics are not seen as two independent and unconnected subjects.

Theoretical framework

The Anthropological Theory of the Didactic (ATD) (Chevallard, 2013) frames the teaching experience reported here. In particular we use the notion of «Study and Research Paths» (SRP) which we explain below.

Study and research paths

Study and Research Paths (SRP) are a didactic device that emerges from the educational paradigm known as “Questioning the world”, in which problematic issues Q are studied, thus emulating the scientific activity, in which open questions motivate knowledge construction and the production of “acceptable” but partial answers.

The inquiry conducted by x on Q opens a route called study and research path. In order to move forward through this path, the inquiry team X has to use the knowledge —connected to the answers R^\diamond , as well as to the works O — hitherto unknown to its members, with which the team must familiarize itself so that it can continue through the path towards the answer R^\heartsuit (Chevallard, 2013, p. 170, our translation)

In this way, the Q issues enable the construction of knowledge. The study of Q must be done by a team of students X , where x is a particular student, oriented by a team of professors Y , or a teacher y , emerging thus a didactic system with the form $(X; Y; Q)$. The operation of this system consists of searching for the answer to Q , through researching the existing answers R^\diamond (knowledge) on O works (books, educational videos, webpages, etc.) that x evaluates and that allow her to generate sub-questions to guide her research. In order to move forward through this path, the inquiry team X has to use the knowledge about both, the R^\diamond answers and the O works.

In sum, a SRP is a didactic device that organizes knowledge in a succession of pairs of questions and answers, where the questions can be related to mathematics and other disciplines. The relevance to the questions is fundamental to overcome the classic paradigm of “visiting the knowledge”, and introduces the new paradigm of “questioning the world”.

Method

The teaching experience reported in this manuscript is part of an on going research study developed primarily by Lina (the first author of this paper) as part of her master studies in mathematics education. Lina is a mathematics teacher in a Mexican high school, while Avenilde and Mario (second and third authors respectively) are her supervisors. It is important to note that during the planning and design phase of the SRP we also had the collaboration of Claudia, a literature teacher who works in the same school as Lina.

Selection of a literary work

First of all, a literary work was selected as a central element of the SRP. Lina and Claudia made the selection, and the selected work was “The Library of Babel” by Jorge Luis Borges. There were several reasons for selecting “The Library of Babel”: first, the story contains mathematical notions relevant to the students’ mathematical education such as the notion of infinity, and some geometric shapes such as hexagons; second, the story can not be understood in its entirety without certain understanding of these mathematical notions²; and third, Borges is one of the Latin American authors that students should read as part of their high school studies.

“The Library of Babel” is a short history where Jorge Luis Borges conceptualizes the universe as an immense library composed of hexagonal galleries. Figure 1 shows an extract from the English version of “The Library of Babel”.

The universe (which others call the Library) is composed of an indefinite and perhaps infinite number of hexagonal galleries, with vast air shafts between, surrounded by very low railings. From any of the hexagons one can see, interminably, the upper and lower floors. The distribution of the galleries is invariable. Twenty shelves, five long shelves per side, cover all the sides except two; their height, which is the distance from floor to ceiling, scarcely exceeds that of a normal bookcase. One of the free sides leads to a narrow hallway which opens onto another gallery, identical to the first and to all the rest. To the left and right of the hallway there are two very small closets. In the first, one may sleep standing up; in the other, satisfy one's fecal necessities. Also through here passes a spiral stairway, which sinks abysmally and soars upwards to remote distances. In the hallway there is a mirror which faithfully duplicates all appearances. Men usually infer from this mirror that the Library is not infinite (if it were, why this illusory duplication?); I prefer to dream that its polished surfaces represent and promise the infinite ... Light is provided by some spherical fruit which bear the name of lamps. There are two, transversally placed, in each hexagon. The light they emit is insufficient, incessant.

Figure 1: An extract from the English version of “The Library of Babel”

² An English version of the text can be found at:
<https://libraryofbabel.info/Borges/libraryofbabel.pdf>

This short story has been analysed and illustrated by literary people, mathematicians, and people who is just interested in these disciplines (see for example <http://libraryofbabel.info/>). This makes available several Works O that, theoretically speaking, the students could draw on to develop the SPR “A library for your community”.

After the selection of the text, the design process of the SPR continued. The final version of the Study and Research Path consisted of three stages: (1) exploring and analysing the text; (2) towards the design of a library; (3) designing a library. Next, we explain each of these stages in more detail, but before that we will describe the characteristics of the context where the SPR was applied, and the characteristics of the participating population.

Context and population sample

The SPR was implemented in a private high school located in a marginalized urban area of Mexico. The implementation was carried out in two groups of high school students from fourth semester (16-17 years), and required four sessions of fifty minutes each one. The groups were organized into teams of between four and five members. In relation to the academic background of the students, at the time of conducting the experimentation they had studied the mathematical subjects of: algebra, geometry, analytical geometry and functions. In the case of subjects related to literature and language, they had taken courses in: reading and writing (two workshops) and literature (two courses).

Stage 1 — Exploring and analysing the text

The purpose of this first stage was for the students to analyse the structure of the library of Babel. For this, a set of triggering questions was posed to the students. Some examples of such questions are:

- What shape does a gallery have?
- Based on your calculations, is the library of Babel infinite or finite?
- What is the difference between an exaggeratedly large amount and infinity?
- Do you think that the library of Babel described in the story could exist. Why? Argue your answer.

Stage 2 — Towards the design of a library

At this stage students were asked to analyse elements for the design of a library in their own community, such as the type of terrain where it could be built, the budget to construct it, the type of design that could be proposed (its geometric

form), the books that could contain, etc. The aim was for a modelling activity to take place, and beyond the ideas of infinity and the literary aspects of the story, to analyse the role of a library in their community and how a project could be proposed to a governmental instance so that it could be materialized. Here it is important to remember that the school where the SRP was implemented is located in a community with few libraries. Some of the questions proposed for this phase are:

- Why are libraries important?
- How many libraries are in your community?
- Are they appropriate to meet the academic needs of the students in your community?
- What characteristics do you consider a library should have in order to meet the needs of your community?

Stage 3 — Designing a library

The aim of the third stage was for the students to build a model of a library for their community. This implies the use of mathematical modelling, to work on forms, scales, geometric design, but also involves analysing their social reality and how it could be modified. We think that through the library mock-ups developed by the students it could be possible to perceive how they imagine and propose a design of a library for their community.

Some results of the implementation of the SRP

In this section of the manuscript we will present some of the results obtained when implementing the SRP. In particular, we present some ideas related to the notion of infinity that the students manifested during the development of the SRP; due to space constraints it was not possible to present in this manuscript results related to the designs of the libraries developed and proposed by the students.

Students' ideas on the notion of infinity

Several ideas on the notion of infinity appeared in the written reports delivered by the teams of students. One of them is that “too big is infinite”. For example, in figure 2 the team number 1 indicates that the library is infinite, but has a finite number of books (answer to question 3). They calculate the number of shelves and books, but assume that since the library is too big, then it is infinite. To argue that its size is infinite, they claim that this is because it resembles knowledge, which they conceive as infinity.

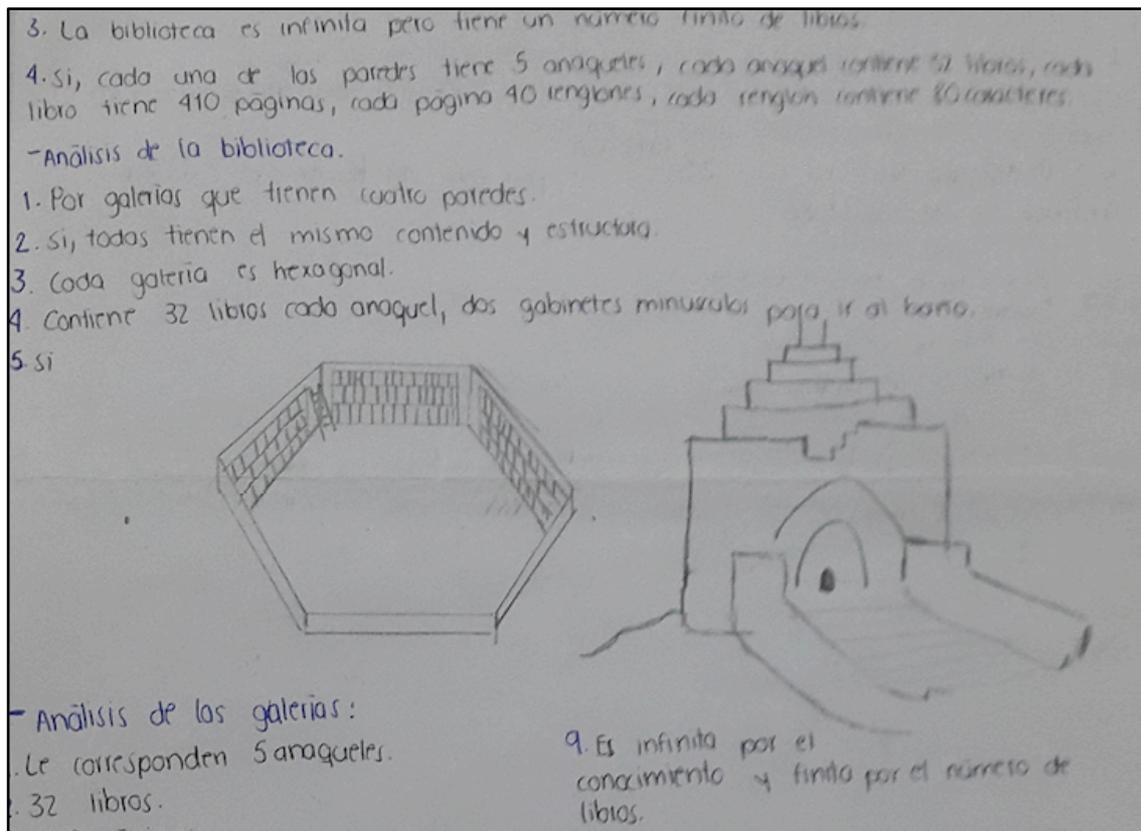


Figure 2: Written report of the team 1, where they express some conceptions on the notion of infinity such as “too big is infinite”

In a similar way, team 2 calculated the number of books, but they claim that it is not possible to calculate the number of galleries; in fact, they represent a gallery as an open structure (see figure 3). Thus, the fact of not being able to calculate the number of galleries makes them assume that the library is infinite, “the countless is infinite” they say.

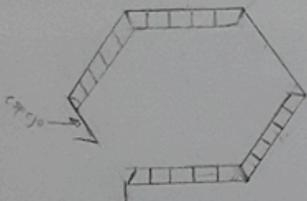
2.1 Por galerías hexagonales y 20 anaqueles, con 2 gabinetes minúsculos.

2.2. Si

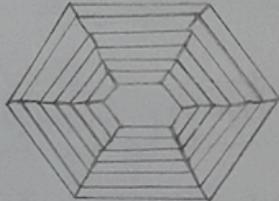
2.3 Hexágono

2.4 anaqueles.

2.5



2.6



3.1 5 anaqueles

3.2 32 libros

3.2.1 410 páginas

3.2.2 40 renglones

3.2.3 80 letras

3.2.4 No se puede

3.2.4.1 Se podría calcular el número de libros en una galería pero no sabemos el núm. de galerías

3.2.4.2 Hay 640 libros x galería pero no sabemos el número de galería

3.2.4.3 Infinita.

Figure 3: Written report of the team 2, where they express some conceptions on the notion of infinity such as “the countless is infinite”

Another team of students (see Figure 4) noted that the library “is infinite because there is no limit”.

63- que el infinito no tiene límite

64- no, porque sería complicado hacer una biblioteca con las características descritas

Figure 4: Written report of another team of students where they claim that the library is infinite because it has no limit

Conclusion

The SRP allows in the first instance to analyse the short story “The Library of Babel” considering both, literary elements (fiction and metaphor) as well as

mathematical elements (the geometric form of the library, its size, the number of galleries, books and pages). This analysis forces us to question the main metaphor in which the universe is viewed as a library, which makes us assume that the too large is infinite. The questions proposed in the first stage of the SRP triggered students' critical thinking in the sense that they questioned the information provided in the short story: can the library actually exist or just corresponds to a fiction? The SRP is a didactic experience that questions reality, on the one hand the reality portrayed in the story, and on the other hand students' reality: is it possible to propose a library for the community where I live? How could I propose it? What would be the benefits? etc. Thus, students are responsible for analysing the story and looking at their context as active citizens that can question, analyse and propose projects for their community.

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THE BEAUTY OF PATTERNS - THE HIDDEN MATHEMATICS OF MUSIC

Hans Peter Nutzinger¹

Abstract

Mathematicians explore patterns and describe them. Rediscovering a known pattern in a yet unknown structure is a great deal of pleasure. Whenever children come in touch with mathematics, we deprive them of exactly this joy when we concentrate our teaching on calculus. At the 10th anniversary MACAS conference 2015 in Schwäbisch Gmünd we described the impact of an interdisciplinary teaching on beliefs. In this paper we show ways to initiate the exploration of patterns - and thereby doing mathematics – when dealing with music.

The problem

“Mathematics is often seen as a very specific subject by students - and even by mathematicians themselves. Although a growing number of subjects include ingredients from mathematics, it is still difficult for both teachers of mathematics and teachers of other subjects to see the use of mathematics in other subjects - partly due to the use of concepts and language.” (Michelsen, 2006, p. 269)

In 2015 we found such use of mathematics in music. Our studies concentrated on the impact of an interdisciplinary approach on beliefs of learners. But though we could find evidence for this impact (Nutzinger, 2015), more theoretical work had to be done before implementing an interdisciplinary teaching approach in school. We concluded:

“Some well-known sources [...] and brand new concepts [...] did not find their way into classes yet. I suggest to reexam their educational value within the outlined context.” (ibid.)

Robichaud & Freiman came to a quite similar conclusion: “Music and mathematics share a rich common heritage which is not being fully explored in educational contexts.” (Robichaud & Freiman, 2015, p. 74)

As no relevant basic research could be found, we undertook an own investigative study in form of a school experiment, which will be presented in this paper. The

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first step was to find a fundamental common concept of mathematics and music that suited our ideas.

“Perhaps the most general aspect of the affinity between mathematic and music might be the perception and articulate study of patterns. Pursuing this agenda within music might encourage children to become intrigued with patterns in other domains as well.” (Bamberger, 2013, p. 324)

So we focused on patterns and learning about patterns as we wanted the “might be” in Bamberger’s statement to read “is”.

To examine the educational value of teaching patterns interdisciplinary the following two problems had to be solved:

1. Do learners actually benefit from teaching it interdisciplinary? (Explorative study)
2. Is there a connection between the concept “pattern” in music and mathematics and can we find an overlap in the two concepts? (Theoretical work)

The concept “pattern” in mathematics and music

Alexander et al. defined the concept domain-independent as follows:

“A pattern describes a problem which occurs over and over in our environment, and then describes the core of the solution to that problem” (Alexander, 1977, p. 247).

One can easily see mathematics in that definition. One part of it is about recurring problems and recurring solutions.

Eichler describes patterns as the core concept of mathematics education at all:

“Numbers and geometrical objects have a variety of relations, which are reflected in [...] [e.g.] patterns. Such structures are the real educational content of mathematics. The children can

- discover and describe the laws in sequences and patterns,
- continue with pre-determined [...] patterns and invent their own [...] patterns,
- identify the consistency of the laws of two [...] patterns [...].”

(Eichler, 2009, translated by the author, p.5)

Furthermore, he expands the concept to the way we solve mathematical problems:

“When we are solving problems: We are not only searching a solution, we are searching for patterns, trying patterns, transforming patterns, keeping successful patterns... That’s our live – we are looking for patterns” (ibid.)

On the one hand, from a mathematical point of view a pattern is therefore a recurring problem with an immanent recurring way of solving it. Finding a law or a solution is a composition of trial and error, transformation of known other solutions and the perpetuation of already found valid solutions.

In music, on the other hand, the concept pattern may not be so obvious at first glance. But patterns build the same structural foundation of music as they do in mathematics. Pulse, pitch, tones, volume become music by relation, by representing patterns:

Rhythm (longer, shorter), pitch (deeper, higher), intervals (bigger, smaller), dynamics (louder, softer), melody, musical structures, lyrics, harmonic relations

For example: The pattern of the lyrics of “Are you sleeping, Brother John”:

Are you sleeping, are you sleeping?

Brother John, Brother John?

Morning bells are ringing, morning bells are ringing

Ding Ding Dong, Ding Ding Dong.

The pattern can be easily found. We replace "Are you sleeping" with "a", "Brother John" with "b" and so on, and get:

a, a, b, b, c, c, d, d

A lot of children come across this song. We enable only a few of them to think about the pattern behind that experience.

Geist claims, that music is children’s first patterning experience (Geist, 2009). We can therefore assume that there are mathematical opportunities in dealing with music which have not been used in class yet.

Patterns in primary school

To solve the second problem, we had to show that it is possible to teach parts of a mathematical concept by musical means or vice versa (see Beckmann, 2003). The fundamental idea of the school experiment was to combine the different views of the concept “pattern” and use a form of representation that suits both of them. “A very obvious starting point in seeking to develop better approaches is to look yet again at the problem of representation.” (Tzonis & White, 1994, p. 449)

Starting from a mathematical point of view we had a look at an excerpt of Geist’s statements about how mathematical learning should take place:

- Mathematics develops from real-life situations in which the child is an active participant.
 - Children learn mathematics through actively engaging their minds in as many different ways as possible.
 - Thinking about relationships, such as bigger, smaller and faster, slower, and especially about pattern relationships, plays a special role in young children's mathematical development.
 - Learning mathematics is a developmental process influenced by the child's physical, social-emotional, and cognitive learning and development [...]
- (Geist, 2009, emphasizes by the author)

According to Geist we find a clear connection to didactics of music in these mathematical competencies (ibid., p.74). Relational expressions as faster and slower play a big role in music education too. Children get active in music classes and, for sure, we can start a learning process which helps to influence the children's physical, social-emotional and cognitive learning and development. Furthermore, recognizing relationships in patterns is a quite common skill in music education.

The connection of music and mathematics can look like this:



Figure 1: Representation of connection between music and mathematics
(Nutzinger, 2017)

This could be the representation of a lot of phenomena. Firstly, it can be the simple discovery that the pattern can be continued with a square and two circles. Clearly a mathematical statement. Secondly, it can be a representation of a 3/4th measure, like a waltz (see Figure 2). The first beat of which is always emphasized a bit and is therefore different to the other two beats in the rhythm. Thirdly, it could be the representation of a 2/4th measure (see Figure 3).



Figure 2: Example of 3/4th measure



Figure 3: Example of 2/4th measure

On more example:



Figure 4: Example of increasing elements (Nutzinger, 2017)

This could be interpreted as an increasing volume of a musical expression (see figure 5) or a speeding up rhythm (see figure 6).



Figure 5: Crescendo



Figure 6: Accelerando or shorter note value

The school experiment

The qualitative explorative study followed the methodology of the “heuristic approach” as suggested by Bräu (2002). As we wanted to observe how children react to our ideas we kept the heuristic framework as small and open as possible.

“A useful heuristic framework for qualitative research, [...], encompasses little empirical content-free concepts.” (Bräu, 2002, p. 245, translated by the author).

So we chose a regular 3rd grade primary school class consisting of 22 children. The class had no special training in music whatsoever. The school was located in a rural region close to the city of Ulm in Baden-Württemberg, Germany. We had 90 minutes of time. The experiment was planned and carried out without the teacher of the class.

We planned to collect data based on the following questions:

1. Are children able to find fitting prepared representations of rhythms?
2. Are children able to produce the mentioned forms of representations?

3. Do children develop mathematical as well as musical competencies?
4. Do children show other unexpected signs of understanding this interdisciplinary approach?

Following Bräu's approach we kept the framework for possible observations completely open.

The school experiment was divided into four parts:

1. Show many patterns simultaneously and let the students find a fitting form of representation for a presented rhythm or musical expression.
2. Let the students play and find fitting patterns with a partner or in small groups.
3. The students develop their own patterns / rhythms and get creative.
4. The students present a collection of their patterns to the class.

The first part was started teacher-centered to allow the children to understand the idea. We showed them several patterns similar to the two examples above. After playing a rhythm we asked which projected pattern fits the sound they just heard.

For the second part we prepared worksheets with several patterns on it. The children should work on these with a partner or in groups of max. 4 people - exactly the same thing as in the first phase but now the students work without the teacher.

The third part sets the learners free to invent their own patterns and rhythms, draw patterns, discuss and share ideas. Afterwards they should present their results.

The rationale of the second, third and fourth part was to see how children react to this task in general and to check whether a development of the stated mathematical skills can be observed.

Results

The students understood the idea immediately. The related pattern to a presented rhythm was found instantly. One child asked to present an own

rhythm which was surprising for us in this early stage of learning a new form of representation. So we started the second phase of the lesson after only 7 minutes.

The students played rhythms and found the representations on the worksheets in small groups. In this phase we already could observe all the supposed mathematical skills: When discussing their solutions students talked about relational characteristics of the chosen pattern. In fact, they used all the problem solving strategies as cited above by Eichler. Furthermore, they were active, used many different ways of representing and playing a pattern. They used relational expressions as bigger, smaller, faster, slower and we observed a developmental process which was influenced by the child's physical, social-emotional, and cognitive learning (Geist, 2009).

The third phase started after 40 minutes. The children produced their own rhythms and drew fitting patterns. Again we could keep the instructions very short, as the children understood instantly.

At this point we asked the head teacher of the class, whether she had practiced something similar with the class before. She did not. She said that she was surprised that the children could do this so fast. She admitted that she herself did not really get the idea so easily when we explained her what we planned to do.

As the children seemed to produce a lot of very good results we decided to keep the presentation phase short. The groups were able to present only one result each in the last 15 minutes. All the groups presented conclusive results. Two of them chose the prior form of showing all their generated patterns then clapping the rhythms and asking for the fitting representation. The others showed one pattern and performed the fitting rhythm to the class, followed by a short explanation.

We want to discuss our observations based on one example drawn by Jonathan:

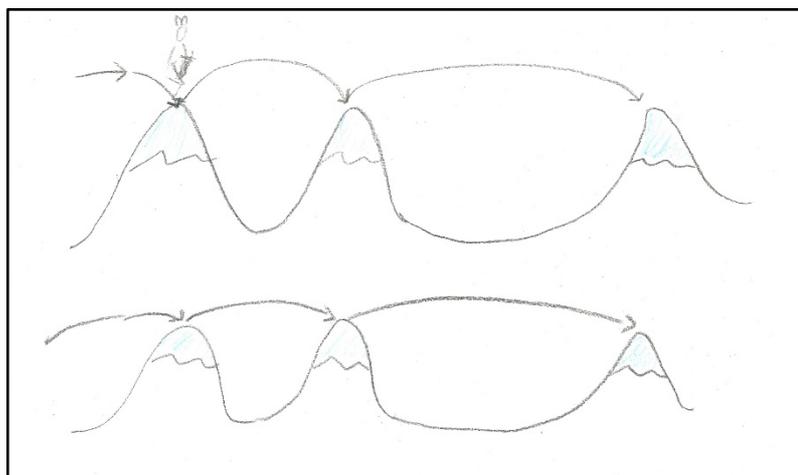


Figure 7: Drawn rhythm (Nutzinger, 2017)

He clapped the following rhythm:



Figure 8: Representation of clapped rhythm

There is a lot of mathematics in his action:

First of all, it is a simple drawing task. The child drew and colored the drawing. By doing so he translated one form of representation of an everyday phenomena (rhythm) into a different form (drawing). He found a fitting model for the length of a tone. His model is the distance between two hills. The length of a line reflects the length / duration of a sound and the gap between sounds.

For sure, his drawing shows a pattern which he generated on his own. He seems to having understood the core principles of patterning. We enable them to see the use of mathematics.

Conclusion

The theoretical work revealed a big overlap in the concepts of pattern in mathematics and music. In fact, the only difference is the way of how patterns are represented. We see a big opportunity to teach the concept in an interdisciplinary way. The school experiment showed that this is possible easily. Children got the idea instantly and showed an advancement of their mathematical and musical abilities and skills. Music is a way of learning about patterning and thereby about mathematics. We therefore adjust Bamberger's cited statement: The most general aspect of the affinity between mathematics and music is the perception and articulate study of patterns.

This school experiment focused on the mathematical concept "pattern" by representing them as rhythms. It is to be assumed that pitch, intervals, melody, musical structures, harmonic relations and dynamics of music offer an even broader insight into mathematical concepts such as pattern, relation and function. This opens a broad field of further scientific exploration.

Finally, this approach shows a possibility of how to prevent students as well as teachers from perceiving mathematics as a "specific subject" with only few connections to other fields. By using the presented teaching method, we enable learners to see the use of mathematics beyond its imaginary borders.

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ON POSSIBLE USE OF TERAHERTZ ELECTROMAGNETIC OSCILLATIONS IN ART EXPERTISE AND PUBLIC ART TECHNOLOGIES

Darya Yeryomka¹

Abstract

Expert evaluation of pieces of art works is an activity which aims to discover compliance of evaluation subjects to authenticity criteria, their correspondence to claimed period of creation and appurtenance to a particular author. The backbone of expert evaluation can be defined as identification and establishment of work of art status, as well as acquisition of primary data about it. Result of experts work is experts report which confirms or denies authenticity of evaluation subject. In modern artistic expertise a range of noncontact physical tools is widely used, in particular, infrared, ultrasonic and X-rays spectroscopy, optical and digital microscopes, acoustic methods, mathematical methods of evaluation results processing. Above mentioned methods have their own limitations and disadvantages. Creation of femtosecond lasers boosted development of terahertz (THz) spectroscopy in time domain – difficult in its physical processes, complicated in its schematic solutions, labour-consuming in processing of measured results. Therefore search of new advanced methods and engineering solutions, which would beat existing ones, still remains a crucial task. Peculiarities and outlook of terahertz emission use in art expertise technologies have been distinguished. We've described design and operation principle of terahertz frequency range spectrometer on basis of source of electromagnetic oscillations – free-electron laser (FEL) with no relativistic energies of particles – orbictron (stands for open resonator, binary comb and electron).

Introduction

Time we live in shows us tight connection of art and science, particularly such areas of science as mathematics and physics. For last decades of XX century vocabulary of arts enriched itself with such terms as computer graphics, computer painting, fractal painting. Probable the above

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mentioned trends of art development were born among mathematicians. Advanced computing machines are capable to create works of art provided that the algorithm (program) is written in mathematics language. Computer art and computer graphics are closely connected to computer mathematics.

Familiarization with advanced art expertise technologies and technologies of public art (which for last decades are being characterized by interdisciplinary approach to solving occurring problems) will help us to acknowledge the synthesis of art and science, particularly art with physics and mathematics.

Technologies of advanced art expertise

Art expertise is an activity directed to detection of work of art compliance to claimed period of creation and belonging to hand of a particular artist. Balance of experts' work is expert report which confirms or denies authenticity of picture, icon, etc. which are subjects to collecting and trading. Art experts' work is based on performing complex iconographic, stylistic, technique and technical expertise of works of art. The experts identify and establish status of work of art and collect initial information about it.

In modern study of art and practice of art expertise different noncontact techniques are used, particularly infrared, ultraviolet and X-ray spectroscopy, acoustic methods, etc. With appearance of impulse femtosecond lasers a THz spectroscopy in time domain started its development.

Physical tools of advanced art expertise is known to the audience that's why I will list techniques only: infrared reflectographics, X-ray fluorescence analysis, infrared spectroscopy with Fourier transformer (FTIR), method of microsampling with help of microscope, expertise with help of ultraviolet light, THz spectroscopy in time domain. Each of above mentioned techniques has its own limitations and disadvantages.

Particularly for THz time domain spectroscopy is characterized by complicated physics, complicated schematics, labour required mathematic processing of measurements results. That's why nowadays the problems related to creation and application of new technical solutions with advance frequency and energy characteristics as well as effective methods based on them are still the ones of high priority in program of exploration of THz frequency range.

Area of space-saving wide-range generators of electromagnetic oscillations of pulse and continuous action with smooth change of output signal frequency are explored poorly. Developed and studied in Institute of radiophysics and electronics of National Academy of Sciences of Ukraine (Kharkiv) space-saving wide-range (0.1-1.5) THz generators of electromagnetic oscillations – LSE-orbitrons with nonrelativistic energies of electrons will find their wide use

particularly for creation of measuring toolkit in technologies of modern art expertise (THz spectrometers for art expertise), In technologies of public art (gas THz spectrometers for monitoring purity of air in urban recreational zones), in technologies of short-range THz radiolocation (application of THz imaging for solving problems of public art).

Results of mathematic modeling of electromagnetic field intensity in resonant system of THz LSE-orbictron showed the fact that mathematics and physics are principium for detection of optimal structure of THz LSE-orbictron as a source of electromagnetic oscillations for THz technologies.

Mathematic modeling of electromagnetic fields in resonant system of THz LSE-orbictron

Art expertise is an activity directed to detection of work of art compliance to claimed period of creation and belonging to hand of a particular artist. Balance of experts' work is expert report which confirms or denies authenticity of picture, icon, etc. which are subjects to collecting and trading. Art experts' work is based on performing complex iconographic, stylistic, technique and technical expertise of works of art. The experts identify and establish status of work of art and collect initial information about it.

In modern study of art and practice of art expertise different noncontact techniques are used, particularly infrared, ultraviolet and X-ray spectroscopy, acoustic methods, etc. With appearance of impulse femtosecond lasers a *THz spectroscopy in time domain* started its development.

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Mathematic modeling of electromagnetic fields in resonant system of THz LSE-orbitron

For detailed description see reference literature, articles [13] to [17].

A schematic representation of the THz resonant system of the FEL orbiter is shown in Fig. 1, a. In the region of the fly-through channel, the y-section of the CCD resonators has the form of a flat double comb (Fig. 1, b). Owing to the small width of the channel $2a$ for the ribbon electron beam, the wave at the working frequency does not propagate along the double comb.

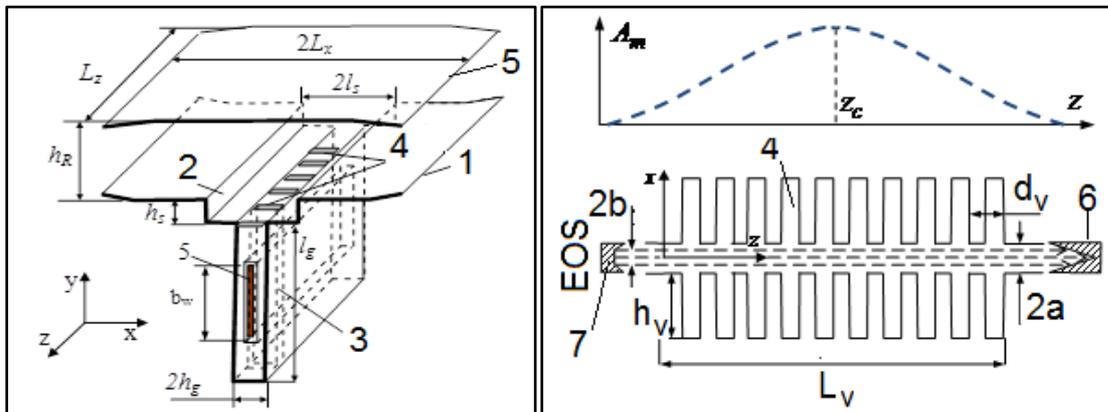


Figure 1. Schematic representation of the resonator system of the orbiter: (a) stationary mirror - 1 OP; Matching resonant groove -2; The periodic structure of the double comb is 3; Half-wave slits - 4 links of a double comb with an OP; Movable mirror - 5; (B) is the Gaussian distribution of the intensity of the high-frequency electromagnetic field over the double comb 3; Half-wave gaps in the connection of a double comb with an OP; The collector of electrons is 6; Emitter of electrons EOS - 7.

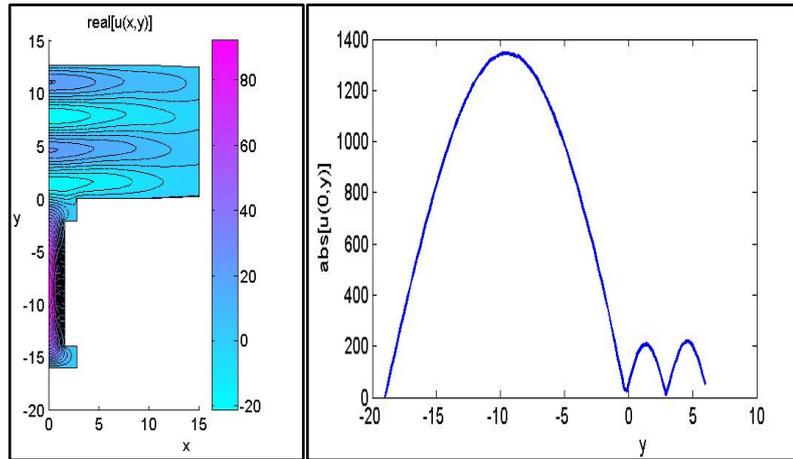


Figure 2. The electrodynamic system of the orbiter.

In such a configuration of the space of interaction of electrons and waves, a more even distribution of the amplitude of the high-frequency field in the zone of the flow of the ribbon electron beam along the width of its ribbon is realized. Fig. 2b shows that the field strength in the volume between the mirrors is more than six times lower than the field strength in the double-comb channel. In the same configuration with the width of the double comb $l_g=17.2029 (3\lambda)$, the field strength in the volume between the mirrors of the OP is six times lower than the field strength in the channel of the double comb in the zone of the flow of the electron beam stream.

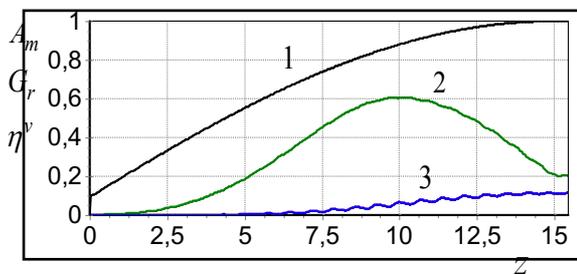


Figure 3, a - integral characteristics. The amplitude of the rf field $A_m = 1$; Functions of electron grouping G_r -2; Generator efficiency η -3

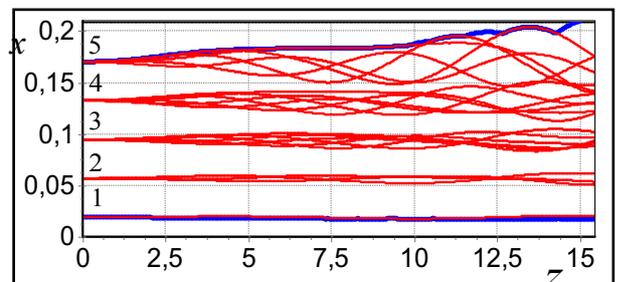


Figure 5, b - trajectory of electrons

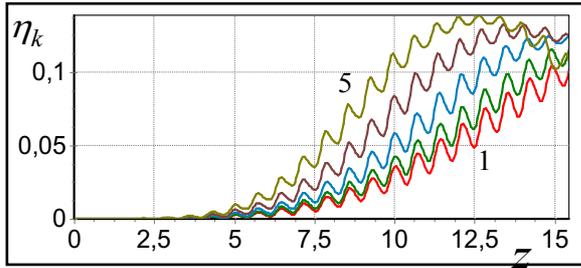


Figure 3, c - Efficiency in layers of tape Electron flow

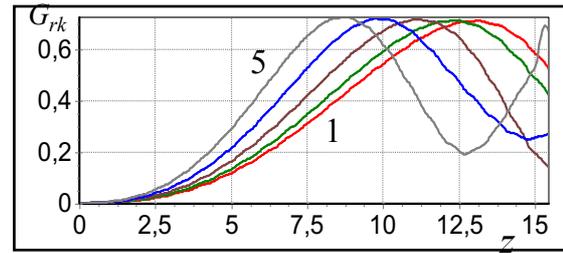


Figure 3, d - grouping functions in layers

Figure 3 shows the characteristics of the interaction of a ribbon electron beam with an electromagnetic high-frequency field when electrons move in a double-comb channel. Along the axis of the double comb. In detail, the results of the mathematical simulation of the THz of an FEL orbictron are described in (Yeryomka et al, 2014).

Spectrometer based on THz LSE-orbictron for art expertise

Quality of radio spectrometers of THz range is preconditioned by two important parameters: sensibility and resolution capability. Resolution capability of radio spectrometer is increased by using emissions of electromagnetic oscillations with high stability, which is achieved by effective methods of stabilization.

Sensibility of THz radio spectrometer greatly depends on Q-value of electrodynamic system which contains the object under examination. One of most perspective course of THz range radio spectrometers is their use in measuring cells systems in the form of high-Q open resonator (OR). Unconventional solution of the cell of highly sensible THz range radiospectrometer is radiospectrometer which measuring cell is integrated into oscillatory circuit – high-Q open resonator (OR) of generating orotron (Rusin & Bogomolov, 1968; Dumesh et al., 1987). If measuring cell of radiospectrometer is a high-Q open resonator of generating orotron then frequency of generator self-oscillations matche resonance frequency of OR to a high precision. As a result, tuning of frequency of orotron self-oscillations doesn't lead to significant change of radio- spectrometer parameters.

Spectrometer based on THz range orbictron can be used for analysis of technological parameters of works of art (pictural art, batik, and woodcarving) and parameters of fabrics (Yeryomka et al., 2011; 2014; Yeryomka, 2013; 2016; 2015). FEL with nonrelativistic energies – orbictron has been studied and produced in A. Usikov Institute Radiophysics and Electronics of National Academy of Sciences of Ukraine (Yeryomka et al., 2011; 2014; Yeryomka, 2013; 2016). Modified orbictron with space between OR mirrors in atmosphere and

vacuum space of electrons interaction with electromagnetic waves has been developed in 2015 (Yeryomka, 2015).

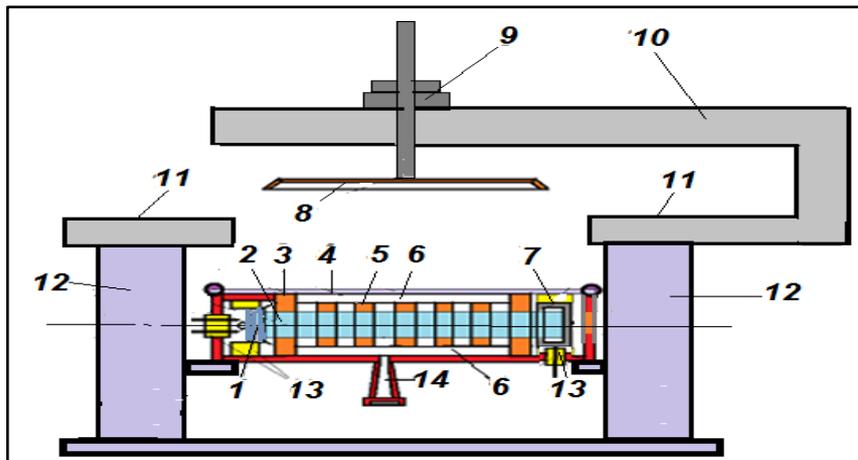


Figure 4 Schematic representation of the basic units of the spectrometer based on a modified THz orbictron: 1 – electron-optical system (EOS), which forms the sheet electron beam (EB) 2; 3 - fixed flat mirror OR; 4 - mica window, evacuated interaction space EP 2 with an electromagnetic field; 5 - diffractive structure – binary comb; 6 - resonance groove in the body of the mirror 3; 6 - electron collector - electron detector; 8 moving through-shaped mirror OR; - mechanism tuning OR resonant frequency; 10 - an arm; 11 - table of the object; 12 - a permanent magnet; 13 – insulators; 14 – waveguide microwave energy output.

Electrodynamic system of modified orbictron provides self-sustained oscillations in frequency range 0.34 to 0.51 THz with possibility of smooth tuning of output signal frequency up to 90 – 40 mW power when operating in continuous mode. Modified orbictron of THz radio spectrometer can operate in pulse mode if required by current task. For schematic representation of THz radio spectrometer based on modified orbictron see Figure 1. The Figure doesn't show power source and computerized results processing scheme FEL-based orbictron is a brand-new vacuum source of THz radiation first time recommended for use in THz radio-spectroscopy. It is reasonable to introduce readers peculiarities of modified orbictron design (Yeryomka et al., 2011; 2014; Yeryomka, 2013; 2016; 2015) describing big difference of the design and characteristics of FEL-based orbictron from design and characteristics of FEL-based orotron which has been used by F. Rusin and his colleagues in spectroscopy on shortwave range (Dumesh et al., 1987).

In FEL-based orotron moving of emitter electrons to collector in parallel to surface of fixed mirror of OR the band electron stream, under which reflective diffraction grating is installed, doesn't provide high energy efficiency of charged particles. In FEL-based orbictron the space of interaction between electrons and

electromagnetic waves in binary comb channel, where intensity of high-frequency field is distributed as to hyperbolic cosine law, provides significant increase of energy efficiency of charged particles. Actual and predicted efficiency of FEL-based orotron of traditional design remain not-too-high. One can mark several factors limiting efficiency of orotrons of traditional design:

- 1) exponential dependence of high-frequency field intensity change at working surface of reflecting diffraction grating on cross-section of band electron stream;
- 2) nonoptimal distribution of amplitude of high-frequency field at symmetrical arrangement of OR diffraction grating (Gaussian distribution);
- 3) relatively weak bond of OR body wave with slow surface wave of diffraction grating, which also acts as slow-wave propagation structure;
- 4) in orotron irregular periodic slow-wave propagation structure – comb, providing realization of electrons optimum phasing and effective extraction of energy from band electron stream, is not used.

In FEL-based orbictron EOS (1) forms sheet electron beam (2) which flows accompanied by magnetic field formed by a magnet (12) from EOS emitter (1) to electrons collector (13) – electronic detector, in channel of binary comb (5) installed in resonant groove (6) made in body of fixed flat mirror (3) of OR. Working surfaces of binary combs (5) are installed in resonant groove (6) in dissymmetric way on planes parallel to OR axis. Side surfaces of binary combs are parallel to working surface of fixed mirror (3). Between working surface of fixed mirror (3) and adjacent side surface of binary comb (5) there is a resonant groove $\lambda_B/4$ deep, which harmonizes space of interaction with volume of space between OR mirrors. Between plane of resonant groove bottom (6) and adjacent side surface of binary comb (5) there is a resonant cavity $\lambda_B/4$ deep, which harmonizes output of high-frequency energy (14) with space of interaction of binary comb (5). EOS (1) with fixed mirror (3) and electrons collector (13) are vacuumized with a help of mica window (4). Fixed mirror (8) of OR is at atmosphere pressure and equipped with mechanism (9) for tuning OR to resonance frequency. Between mirrors 3 and 8 of OR there is a working surface of the bench, on which the object under examination is installed.

Generation of electromagnetic emission in FEL-based modified orbictron is realized in zones of working voltage in e few volts. The required frequency is provided by selecting axial modes $00n$ of OR, by moving mobile mirror (8) with a help of the mechanism (9) along the OR axis. At this operating wavelength is

set to $\lambda_m \approx 2l/n$, where l is a distance between mirrors 3 and 8. Distances between mirrors 3 and 8 along the OR axis usually are not big ($m = 20 \div 40$), there is no overlap between modes. FEL-based orbictron operates in single mode. Selection of modes is made with help of mechanism (9) for moving mobile mirror (8) with U-shaped working surface. Frequency-energetic parameters of FEL-based orbictron of THz range: frequency range of generated oscillations $0.36 \div 0.48$ THz, power of output signal in continuous operation mode $50 \div 100$ mW (if necessary FEL-based orbictron can operate in impulse mode); bandwidth 15 MHz; operating voltage $2 \div 4$ kV; operating current $I_a \approx 100$ mA; operating magnetic field $b = 0.8$ T; magnet gap 36 mm. OR Q-value $\approx 7 \cdot 10^3$; operating voltage instability $1 \approx 10$ mV.

Fast response time and high sensibility of electronic detector allows registering high-speed process.

Gas spectrometer based on THz LSE-orbictron for Public Art of urban recreational zones

A high-sensitivity gas spectrometer based² on the THz FEL orbiter will provide control of clean air in public art recreation areas of cities.

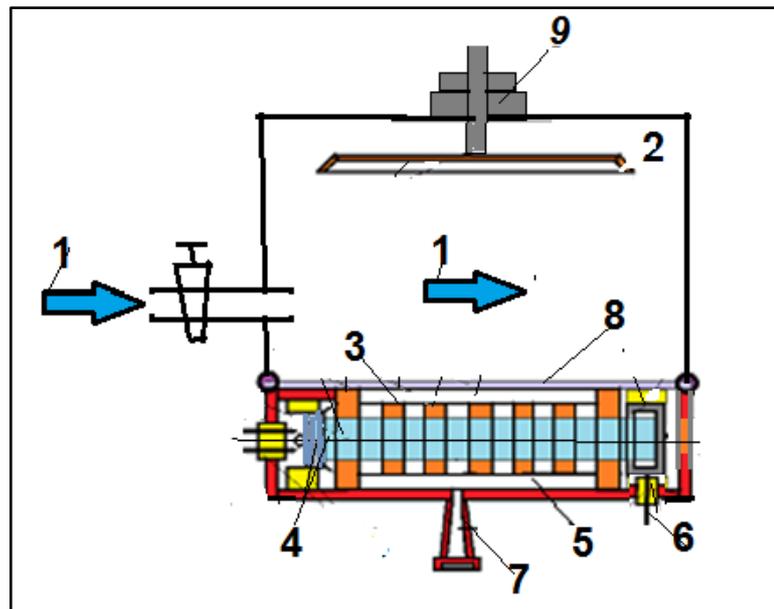


Figure 5: Schematic representation of a gas spectrometer based on the THz FEL orbictron (THz FEL-orbictron) based on THz: 1 - the investigated gas (air), 2, 3 - open resonator (OR) THz FEL-orbiter, 4 - An electron gun forming an electron beam is formed in a fixed mirror 3 of the OP, a binary comb 5, an electron collector

² A gas spectrometer at the THz FEL-orbitron is described in Yeryomka V.D., Yeryomka D.V., Inventor's application to Ukrpatent apply for technical solution " GasTHz Spectrometer based on an orbictron" (May, 8, 2017).

6, a high-frequency energy output THz of an FEL orbiter 7, A dielectric plate 8 that vacuums the interaction space of electrons and waves in the FEL orbiter (plate 8 is transparent to microwaves), the resonant frequency tuning mechanism 9 of the open resonator of the OR FEL of the orbiter. The processing of the results of gas spectroscopic measurements is carried out using computer programs.

THz radars and THz imaging

In order to get artistic solution of figural forms in open or enclosed urban spaces, one needs to possess combination of knowledge and skills of basic architectural composition, development and methods of monumental art. There are several creative groups (teams of specialists) who study state of urban landscapes and solve problems of provision of urban amenities using methods of public art, e.g. CEC ARTLink (New York)³, V-A-C Fund (Russia)⁴, Project for Public Spaces, concept "Power of 10"(USA, Chicago)⁵, Viar Estudio Arquitectura⁶, Drozdov & Partners⁶, etc. I perform art studies related to studying Principles and methods of development of open space urban areas and recreational zones using methods of public art and technique of monumental art together with A. Usikov Institute for Radiophysics and Electronics National Academy of Sciences of Ukraine (Kharkiv).

Under epoch of population mobility complex design and aestheticization of urban streets and parks acting as "beautiful places" are important for activation of economic growth and tourism. As Peter Kageyama, founder of "Creative cities" Summit (USA), says: "There is no need any more to create places which can be only functional and safe. Our aspirations shall be as high and as great as our economic goals, as they are connected to each other"⁵.

³ <http://www.cecartslink.org/>

⁴ http://www.v-a-c.ru/ru/special_project/

⁵ <https://www.pps.org/reference/streets-as-places/>

⁶ <http://projects.viarestudio.com/tagged/Qianhai+Monument+Landscape+Island?og=1dov-partners.com/html/>



The new pavilion gives a breath of life to the square, while leaving its surface free.

Drozdov & Partners.

In nearest future THz radars of short-range radiolocator will find application in solving problems of Public art and organization of landscapes of urban recreational areas. Wide range THz FEL-orbictron will find application in transmitters of THz short-range radars.

Also a method of 3D-scanning (structured light scan) are capable of 99% of that most of artists strive to achieve by more complex efforts. For example of a method in order to create piece of public art was "Liberty Bell" project (2006) and Play-Doh (2014) by Jeff Koons⁷ was 3D-scanning occupy internal and external volumes of each detail of geometrically complicated object and gives possibility to create unique analogue.

⁷ <https://qz.com/235891/the-science-behind-the-art-of-jeff-koons/>



I offer THz radar based on broadband THz FEL-orbictron combined with 3D scanning technologies and computer modelling as a technology for creation of artistically structured urban areas in context of public art. Computer modelling, taking into account architectural design and monumental decoration on the principle of golden ratio, optimizes information obtained with help of THz radar.

Results of 3D scanning of urban landscape with help of THz radar reflect peculiarities of urban landscape and bring to light harmonically balanced and unbalanced voids and saturations of the area. They can formulate target of public art artists, solution of which must be unique plastic forms which eliminate all deviations violating full harmony of area outline in urban landscape and architectural space. Such technical possibilities will promote realization of art projects in urban area directed to creation of spaces where art and city interact in context of cultural and historical heritage.

Development of public art in 1960-1970 resulted in limited inclusion of volumetric plastics of objects into three-dimensional environment. Synthesis of form was achieved not with architecture but with natural environment, park (Krauss, 2003).

Modernist period for sculpture (plastic form) is characterized with phenomenon of place loss: a monument is simlized to an abstraction, designator or pedestal functionally devoid of place (Krauss, 2003, p. 278).

Krause Knight claims that this to a large extent is explained by the fact that places of installation were not prepared for modernist monuments.



Works by Alexander Calder "La Grande Vitesse" (1969) installed in Grand Rapids, Michigan, USA⁸.

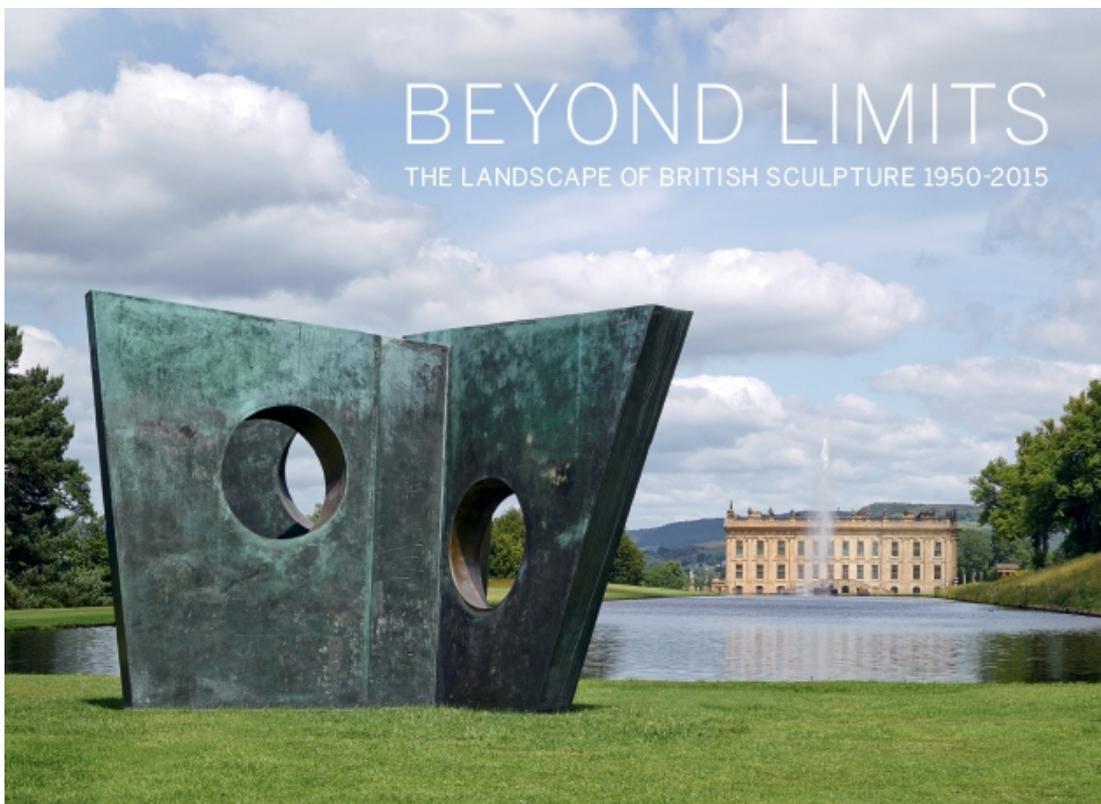


Richard Serra «Tilted Arc», 1981–1989

⁸ Kwon Mivon, On four strategies of public art development
<https://special.theoryandpractice.ru/miwon-kwon>



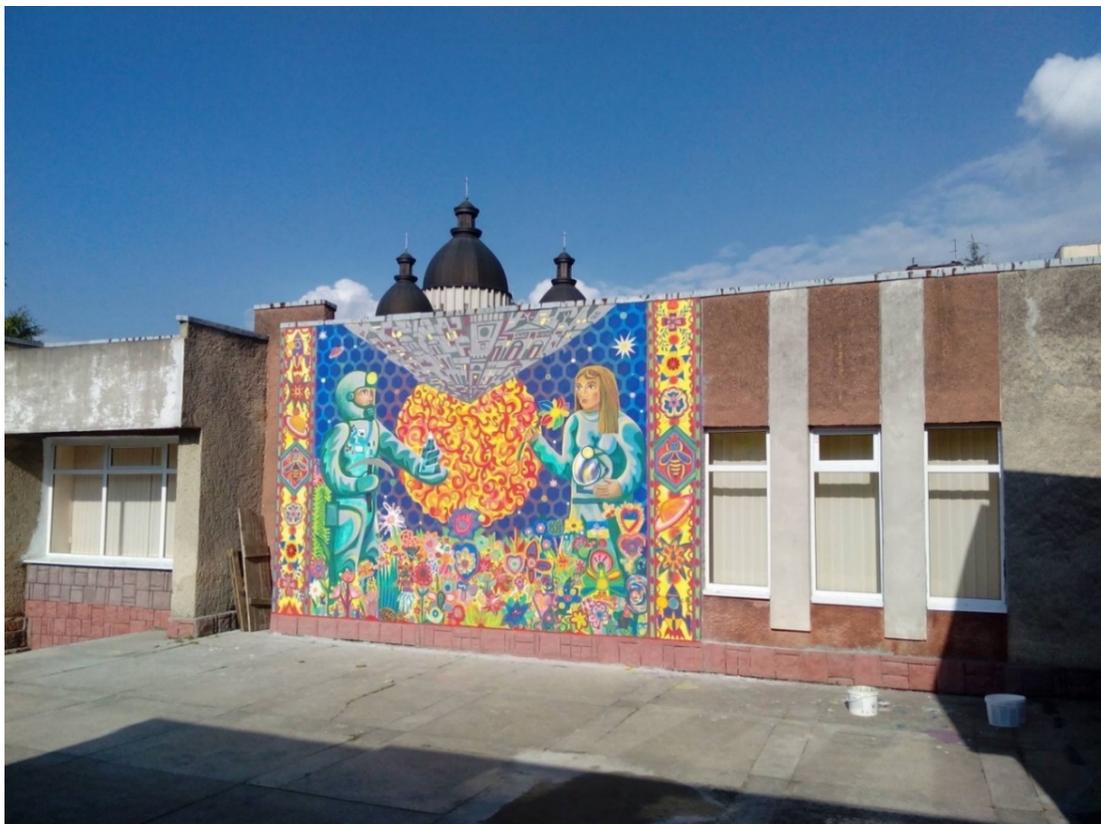
Thomas Schütte, United Enemies, Public-Art Fund Central Park.



County Debishir. England.



Fountain. Geneva. Switzerland.



Mural « Space Cathedrals». Ukraine, Lviv. Young Kharkiv artists.



Jorge Luis Rodriguez, Growth. The first sculpture in the framework of the Percent-for-Art in New York.



New-York, mural.



Lang Baumann, Moscow 2011.



Yeryomka Daria, "Complex monumental organization of the recreational zone in the Kharkiv city", master's degree project, 2011.

Wide-range THz FEL-orbitrons will find use in transmitters of near-radar THz radars. THz radars of near radar will find application in solving problems. Public art and the organization of landscapes of zones of rest of cities in the near future.

Conclusion

Use of mathematical and physical principles during development and creation of non-contact physical toolkit for art expert examination results in reduction of financial costs and material losses, as well as in creation of new effective devices. FEL-based orbictron of THz range provides generation of electromagnetic emission both in continuous and pulse modes with power of output signal sufficient for examination of process characteristics of images of works of art (paintings, icons, batik, and wood-carving) as well as characteristics of fabrics. Frequency stability of its output signal allows using phasing methods for examination of canvas layers in three-dimensional arrangement, i.e. without cutting the canvas.

In our opinion FEL-based THz spectrometer will find extensive use as an effective measuring device for works of art expertise, for gas THz spectroscopy and for THz radar transmitters. Gas THz spectrometers with high sensitivity will find extensive use for solving problems related to control and provision of ecologically clean air in recreational zones of cities. In subsequent years THz technologies will also find extensive use for engineering and realization of public art in urban landscapes.

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“MORE GIFTED THAN GIFTED” – MATHEMATICAL COMMUNICATION COMPETENCY AS AN INDICATOR FOR GIFTEDNESS

Peter Weng¹ · Uffe Thomas Jankvist²

Abstract

This paper addresses mathematically gifted students and their capacity to express themselves in regard to their mathematical problem tackling when in dialogue with a teacher. Through a case study of a 3rd grade student – Celeste – it is argued that a well-developed communication competency may also be an indicator of mathematical giftedness. In doing so, attention is drawn to the problem of many teachers not being equipped for engaging in dialogue with such students and thus not being able to facilitate their mathematical learning in a productive and efficient manner. Drawing on the theoretical constructs of the Danish mathematics competencies framework (KOM) and the so-called Inquiry Co-operation Model (IC-Model), we propose a refocusing of the didactical triangle as a potential way of overcoming this problem.

Introduction

Traditional indicators of “mathematical giftedness” within the mathematics education research literature include: unusual curiosity; ability to understand and apply ideas quickly; ability to see patterns; abstract thinking; being creative and persistent (Stepanek, 1999). In addition to these, Ernest mentions: mathematical self-confidence; self-efficacy; attitudes; and motivation (Ernest, 2011). All of these indicators are connected by the student’s ability to communicate his or her thinking to the teacher, or others, in a dialogue about mathematics. Despite the fact that communication must be seen as central in relation to the above indicators, we are not aware of any research focusing specifically on mathematical communication in relation to mathematical giftedness. In this paper we shall do so.

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More precisely, we focus on a student's communication competency when solving problems and engaging in mathematical dialogues. We provide a case study of a 3rd grade student, Celeste, who besides being gifted also possesses a very well-developed communication competency. Celeste was originally spotted as part of the Danish TMTM (*Tidlig Matematikindsats Til Marginalgrupper* – Early Mathematics Intervention Program for Marginal Groups) project, an intervention study from 2015 and on, concerning as well low as high performers in 2nd grade school mathematics. In a period of twelve weeks, two low and two high performers took part in 48 individual lessons with a specially educated teacher. These teachers had a 30 hours course to learn how to use specially designed material (Lindenskov & Weng, 2013) as a background for starting dialogues with low and high performers about belief and attitudes related to mathematics. Involved in the TMTM study were 82 such specially educated mathematics teachers and 281 pupils in the beginning of 2nd grade from 41 different schools as well as a number of associated researchers. From the observations that the researchers did in the TMTM study of how the teachers were acting in relation to the pupils during the interventions, we noticed Celeste³. She was one of the very few high performers among the approximately 150 pupils in TMTM who were able – and motivated – to go into a learning dialogue with the teacher.

During our interview-based interventions with Celeste we have come to realize the importance of her mathematical communication competency in relation to her problem tackling as well as her mathematical reasoning. Time and again, we have witnessed that our attention to her way of communicating her thoughts concerning the mathematical problems presented to her have resulted in activating her other mathematical competencies, this leading to further mathematical understanding on her behalf. Clearly this only happens, if Celeste is communicated with. Hence, the underlying aim of this paper – besides pointing to students' communication competency as an indicator of giftedness – is to argue for the importance of mathematics teachers being observant to and actively facilitating gifted students' way of communicating their mathematical thinking.

On mathematical giftedness

Research studies on mathematically gifted students are much less in number than research on challenged students, i.e. students with mathematics-related learning difficulties. This is also stated by Szabo (2017), who has conducted a review of some 180 reports on research studies related to mathematically gifted students from 1953 to 2014 – this taking into account that no prevalent definition of

³ Link: <http://www.egmontfonden.dk/Upload/Egmontfondendk/PDF-filer/TMTM2014%20afrapportering%208%20dec%202016.pdf>

mathematically giftedness exists (but we shall not go into this discussion in the present paper). Szabo, finds that the 180 research studies may be divided into five relatively different categories:

- gifted students' *performances* in mathematics,
- gifted students' *social situation* in school as well as *gender differences*,
- *teachers' perception* of gifted students in mathematics class,
- *definition and identification* of mathematically giftedness as well as national programs for mathematically gifted students,
- gifted students' *motivation and cognitive capacity*. (ibid., pp. 26-27)

As for the content of the research studies in relation to gifted student's performances, Szabo identifies four:

- gifted students in *heterogeneous* mathematics classes,
- *acceleration* of gifted students (through advanced courses, etc.),
- grouping gifted students according their *level*,
- gifted students' attitudes to different work methods. (ibid., p. 27)

In the description of these categories and areas there is no focusing on dialogue and communication between teacher and student. This supports our initial statements in this paper, i.e. that focus on these two aspects of high performers in mathematics education have been somewhat “neglected” in the research literature – unlike the research literature on low performers where several studies focussing on dialogue and communication exist. Another aspect which is worth drawing attention to is that when it actually comes to dialogues with mathematically gifted students, then the interlocutors are usually not student and teacher, but student and parent. To some extent this is confirmed by the relatively high number of texts concerning effective communication between gifted student and parent (e.g. Smutny, 2015).

The Danish mathematics competency framework

The Danish competencies framework (KOM) defines a mathematical competency as (an individual's) “...well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge” (Niss & Højgaard, 2011, p. 49). The framework consists of eight distinct, yet mutually related competencies, which can neither be possessed nor developed in isolation from one another. These are the competencies of mathematical: thinking; problem tackling; modelling; reasoning; representation; symbol and formalism; aids and tools; and communication. Each of the eight competencies has what might be thought of as a producing and an analysing side. Here we briefly outline the three

competencies which we consider most important for the study addressed in this paper, i.e. those of communication, problem tackling, and reasoning.

The *communication competency*, firstly, consists of being able to study and interpret others' written, oral, or visual mathematical expressions or texts. Secondly, it consists of the capability to express oneself in different ways and at different theoretical or technical levels about mathematical matters, again either in written form, orally, or visually. Since written, oral, or visual communication in and with mathematics make use of various representations, the communication competency is closely connected to the representation competency – and also the symbol and formalism competency, since such communication often relies on mathematical symbols and terms. However, the communication competency goes further than the others since the communication happens between the sender and receiver, and their situations, backgrounds and prerequisites need to be taken into account for communication in the same way that purpose, message and media must.

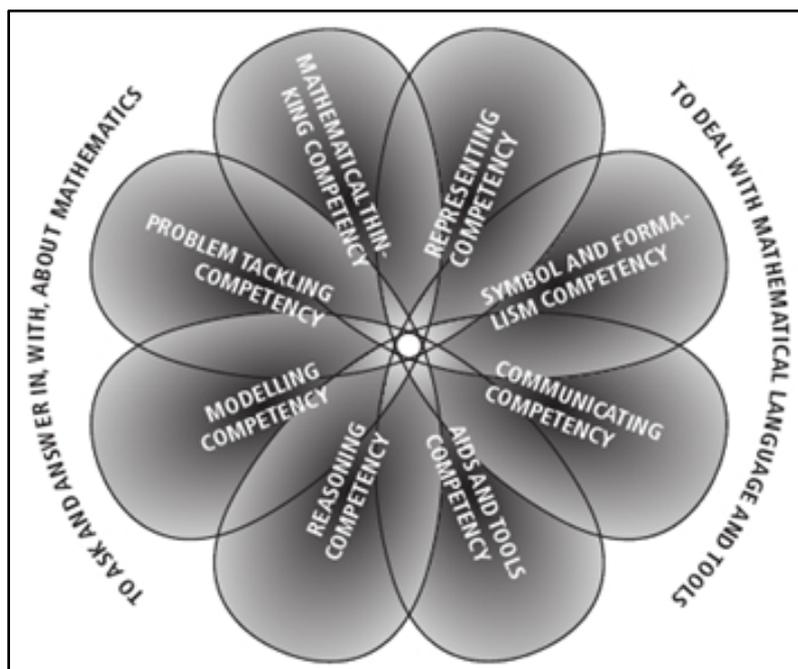


Figure 1. The KOM flower illustrating the overlap of the eight mathematical competencies around a "centre of gravity", a non-empty intersection of the eight competencies (Niss & Højgaard, 2011, p. 51)

The *problem tackling competency* involves the ability to detect, formulate, delimitate, and specify different kinds of mathematical problems, pure and applied both, as well as being able to solve mathematical problems in their already formulated form, whether posed by oneself or by others. The important thing to notice about this competency is that the word "problem" is relative to the person who is trying to solve it, what is a routine task for one person may be

a problem for another and the other way around. The *reasoning competency* consists, first, of the ability to follow and assess mathematical reasoning, i.e. a chain of arguments put forward in support of a claim. Besides this it also concerns understanding the basic ideas of a proof, and when a chain of arguments does or does not constitute a proof, including also being able to understand the role and logic of a counter example. Secondly, this competency consists of the ability to actually devise, carry out and explain (valid) mathematical reasoning, in particular proofs.

Dialogue in learning mathematics – the “IC Model”

As further theoretical background for working with communication between teacher and a mathematically gifted student, we draw on aspects of Skovsmose’s and Alrø’s (2002) framework of *Inquiry Co-operation Model* – referred to as the “IC-Model”. The IC-Model deals explicitly with the type of communication taking place between people working on mathematical tasks. Skovsmose and Alrø state: “Not any kind of communication can be characterised as a dialogue. In general terms, we describe a dialogue as an inquiry process which includes an exploration of participant perspectives as well as willingness to suspend one’s preunderstandings – at least in a moment” (ibid., p. 15). The characteristics of a dialogue as an inquiry process come from the view of mathematics as landscapes of investigation. The IC-Model points out a cluster of eight elements that the teacher should be aware of when in dialogue with (gifted) students ready to go into an inquiry process in mathematical problem tackling. These are:

- *[G]etting in contact* involves inquiring questions, paying attention, tag questions, mutual confirmation, support and humour.
- *Locating* has been specified with the clues of inquiring, wondering, widening and clarifying questions, zooming in, check-questions, examining possibilities and hypothetical questions.
- *Identifying* involves questions of explanation and justification and crystallising mathematical ideas.
- *Advocating* is crucial to the particular trying out of possible justifications, and it is closely related to arguing and considering.
- *Thinking aloud* often occurs as hypothetical questions and expression of thoughts and feelings.
- *Reformulating* can occur as paraphrasing, completing of utterances and staying in contact.

- *Challenging* can be made through hypothetical questions, examining new possibilities, clarifying perspectives, and it can be a turning point of investigation.
- *Evaluating* implies constructive feedback, support and critique. (ibid., p. 110, bulleting added by us)

All these concepts are vital for a teacher to possess and pay attention to as part of his or her theoretical background to be qualified in taking part in a dialogue with students about mathematical issues – but it appears to be even more so important when it comes to mathematically gifted students.

Analysing Celeste's solution to a PISA task

The following task is a PISA task released after PISA 2012. Among Danish 15-year olds (typically 9th grade), 31.6% did not get this task correct (Lindenskov & Jankvist, 2013).

Question: You are making your own dressing for a salad. Here is a recipe for 100 millilitres (mL) of dressing. How many millilitres (mL) of salad oil do you need to make 150 mL of this dressing? Justify your answer. [PISA item PM924Q02]

Salad oil:	60 mL
Vinegar:	30 mL
Soy sause:	10 mL

In a study of 315 upper secondary school students (10th – 12th grade), Jankvist and Niss (in review) found that the majority of those students who got the answer wrong (only around 10%) did so because they were unable to decode the meaning of “this dressing”, i.e. that when scaling up, one should maintain the proportions between the ingredients. As such this task is one that tests students' capability to reason proportionally. At the same time, it also tests simple arithmetic skills; which exactly of course depending on whether the student applies an additive or a multiplicative strategy. But let us take a look at how Celeste tackled the task.

Uffe: Should we try the one with the salad dressing?
 ... Have you tried baking a cake at home, and
 then you had to change the recipe? Maybe you
 wanted to double the size or something.

Celeste: I've tried that quite often.

- Uffe: This is the same. We’re just making a dressing.
[reads aloud the question]
- Celeste: Well, without... if you take out the others, so it was only the salad oil... or...
- Uffe: Well, it’s... [points to the table] Here is a recipe, you see that. It adds up to 100 mL, because all together these give 100. So, we want you to make a dressing of...
- Celeste: 150 mL.
- Uffe: So, how much do you have to put in of each [ingredient]? If it is to taste the same?
- Celeste: Well...
- Uffe: You can write over here, if you want. [draws one more column of the table] How do you figure this out?
- Celeste: Maybe you could... [...]

Using the IC-Model, we see that Celeste pays attention to the problem formulation, she asks inquiring and clarifying questions (about the composition of the ingredients), which are the elements of “getting in contact” and “locating”.

- Peter: Try to write 100 there. [points to the original column in the table] And then put 150 at the bottom of the one you are to do now. [points to the new column]
- Celeste: So you should... if we only are to find out that it should give 50, then we have this plus this, right? [points to the two columns]
- Uffe: Yeah, what do you mean?
- Celeste: Well, if we have our 100 here. If we then find out somehow a way for it to give 50 instead of 100... then you could add it together with the other result you had... and then...
- Uffe: Okay, we can do it like that. So, you would prefer that we have 50 over here instead of the 150 [points to the new column]... that is to find out how we make the 50. And how do we do that?

- Celeste: Well, we could... maybe we could minus... or use the half of all the things. That would be it, if it has to add up to 100 in total.
- Peter: That was pretty smart.
- Uffe: Yes, it's a good idea. Try to write it down. You can write over there. [points to the new column]
- Celeste: [plots half of the original ingredients into the new column – cf. figure 2]
- Uffe: Should we make yet a column over here for the 150? [draws yet a column]
- Celeste: Yes, okay! So, we say... So, this is 90, right. And this is 45... and then it gives 150.
- Peter: That is super.
- Uffe: Yes, this was very good indeed.

Spørgsmål 10

Du er ved at lave din egen dressing til en salat. Her er en opskrift på 100 milliliter (ml) dressing.

Salatolie	60 ml	30	90
Eddike	30 ml	15	45
Soyasauce	10 ml	5	15
	100	50	150

Hvor mange ml salatolie skal du bruge for at lave 150 ml af denne dressing? Begrund dit svar.

Figure 2. Celeste's solution to the PISA task.

In this next part of the dialogue we witness that Celeste considers different ways of tackling the problem, which is part of “advocating” and she does so while “thinking aloud”. Possibly Celeste could have chosen a more multiplicative proportional model, but in this case she chooses a more additive approach, i.e. taking half the amount of the original ingredients and adding to the full amount – all aspects of the element of “identifying” in the IC-Model. Throughout the entire dialogue Celeste continuously stays in contact with us as interlocutors and she is open to constructive criticism and support in relation to her problem solving – “reformulating” and “evaluating”.

From a mathematical competencies point of view, Celeste of course activates her problem tackling competency as well as aspects of her reasoning competency. But above all, we are interested in the activation of her mathematical communication competency. She is able to interpret the interlocutors' mathematical statements and expressions. Also, she is able to express herself and

her reasoning, using both words as well as by writing in the table (table 2). The dialogue of course happens between senders (us) and receiver (Celeste) with rather different backgrounds and prerequisites. Still, this asymmetry seems not to bother Celeste, rather she seems to profit from it in the sense that she is able to gain the information needed from us in order for her to make sense of the problem and eventually target a solution.

Concluding discussion

As in the case of Celeste, “curiosity” is the predominant characteristic of students who are mathematically gifted. This recurs in both interviews with parents and teachers as well as the majority of researchers who describe characteristics of mathematically gifted students (Petterson, 2011). But it is not this curiosity which is in focus when teachers are asked to identify such students. What usually triggers a teacher’s attention as an indicator of mathematically giftedness is the ability to work very fast, finish before one’s classmates, provide fast and correct answers to the questions posed by the teacher in class, work independently, and more or less effortlessly do well on tests. What Petterson (2011) and others point to is that the mathematically gifted students might very well share these characteristics, but that they far from make up a homogeneous group. One of the characteristics that make them into an inhomogeneous group is their very different capabilities when it comes to expressing and communicating their own mathematical thinking. Not least in early primary school there are students who can come up with the correct result without being able to explain their line of thinking in arriving at it. There are also students who can come up with algorithms or even use tools and aids in obtaining their answers, but generally their arguments are incoherent. Only very few of these young students are able to formulate and generalize their reasoning in the form of a step-by-step account of their solution procedure and the ideas and associations they have encountered during this. By putting more focus on the student’s communication competency as part of the solution procedure, the teacher may gain an insight into the individually gifted student’s thinking and in that way obtain an instrument to identify this student’s special need for support as a student in the inhomogeneous group of mathematically gifted students.

We should like to point to two observations. Firstly, in several studies, mathematics teachers express their own lack of competences to communicate with mathematically gifted students and that they often feel helpless in their attempts to do so. In her study, Petterson (2011) found that: “There are even teachers who are nervous that they do not suffice for the students who find the subject particularly easy, who do not have sufficient competence, and who do not receive support from the school management.” (p. 6, our translation from

Swedish). The observation that teachers find it difficult to have dialogues with mathematically gifted students is also an observation in the TMTM study. Secondly, in our literature searches we have not been able to find any studies specifically focusing on the communication between mathematically gifted students and teachers – only between gifted children and their parents. Taken together, these two observations suggest that it makes up a real problem for many mathematics teachers to teach mathematically gifted students, because the teachers are not able to communicate with such students in ways that support their learning.

As mentioned, all the elements of the IC-Model are vital for a teacher to possess as part of his or her theoretical background to be qualified in taking part in a dialogue with students about mathematical issues – but it appears to be even more important when it comes to communicating with the mathematically gifted students. Hence, in the way that Skovsmose and Alrø (2002) talk about mathematical dialogue and learning, we find theoretical perspectives which can act as guidelines for the way in which teachers should be educated to become *competent* to engage in dialogues with mathematically gifted students. As a way of potentially bringing this into practice, we suggest a refocusing of “the didactical triangle” in relation to dialogue as shown below.

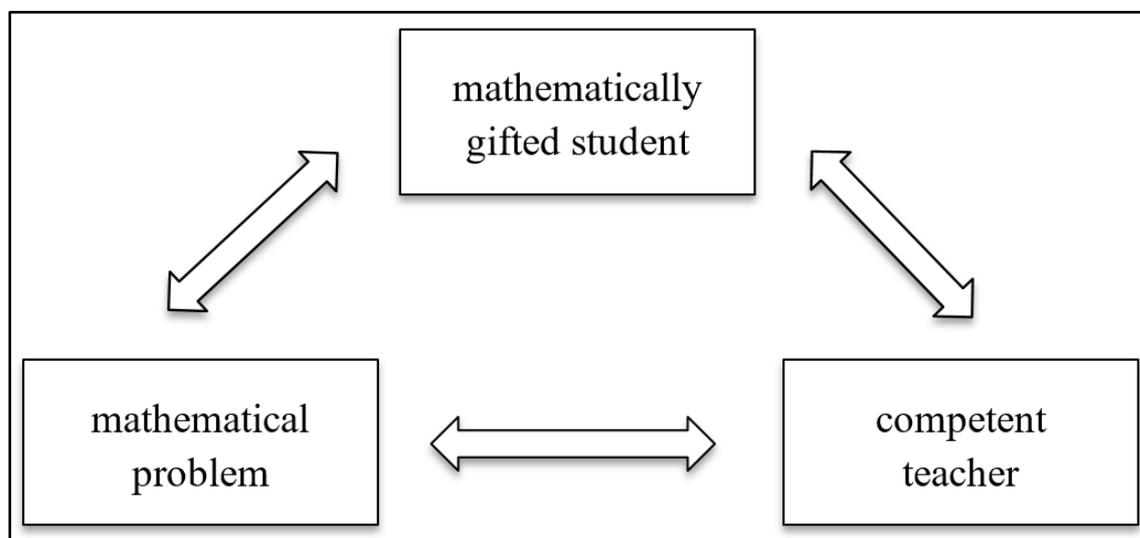


Figure 3. A refocusing of the didactical triangle through problem tackling.

When we usually think about the didactical triangle and the asymmetric relationship between teacher and student, it is the teacher who possesses the knowledge and who is in charge of steering the dialogue. But when the dialogue is between a teacher and a mathematically gifted student, then the opposite situation is not unusual. Although such students may be few in number in early primary mathematics education, they do exist as we have seen from the example

with Celeste. These students belong to a group within the group – they are in a sense “more gifted than gifted”.

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SEVEN KEYS TO EXPERIENCING AESTHETIC ASPECTS OF MATHEMATICS – EXEMPLIFIED IN A DANISH EARLY INTERVENTION PROGRAMME FOR HIGH AND LOW PERFORMERS

Lena Lindenskov¹

Abstract

My aim is to combine and exemplify ideas concerning aesthetic aspects of mathematics research and mathematics learning from leading scholars. First, ideas from Henri Poincaré and Nicolas Bourbaki about mathematics: what constitutes mathematical work and how best to organise it. Second, ideas from André Mack and Nathalie Sinclair about mathematics learning: how beauty, surprises, and meaningfulness are aesthetic motives for mathematics learning. These ideas are the main theoretical background for the ‘Seven keys’ model, which is then presented. The Danish national aims, goals and teaching guidelines for primary and lower secondary school are then analysed using the seven keys model. Some grade 2 practice situations are being described. They point at how practice mirrors the seven key model and the national steering documents. My hope is to inspire teachers and learners widely to promote creativity in school mathematics, if not on a daily basis, then at least more than few times a year.

Theory

The references I build upon are from mathematics scholars and mathematics education scholars. Jules Henri Poincaré showed how aesthetic aims and instrumental aims are inseparable and interwoven in mathematics research (1898, 1903, 1904). Henri Bourbaki constantly played with identity when writing new textbooks and research papers (Bourbaki, 1950) (Aczel, 2006). André Mack analysed educational mathematical practice and artefacts as aesthetic constructs. He concluded that students should know the true motives behind mathematical concepts and investigations, they should have opportunities to experience mathematics as wondrous, and they should see mathematical constructs and ideas as meaningful and of aesthetic value. Therefore, in mathematical curricula and standards the development of students’ imagination ought to be highly

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prioritised (Mack, 2006). Nathalie Sinclair analysed how aesthetics function in mathematics learning and research. She found aesthetics affected motivation, the generation of new ideas and the linking between old and new ideas. Besides, she found aesthetics in use for the sake of evaluating mathematical models, signs and processes (Sinclair 2006, 2008). Obviously, the evaluative function may depend on the actual socio-mathematical norms for the learning setting.

The ‘Seven keys’ model

Reading these theoretical scholars’ work on mathematics research and school mathematics has inspired me to focus in depth on how creativity are required and promoted through mathematical learning in school.

From Jules Henri Poincaré I generalised the idea that aesthetic aims and instrumental aims are inseparable and interwoven to fit also teaching and learning in school. This means that aesthetic aspects and creativity are not just seldomly relevant; they are truly relevant in all examples, sentences, calculations and arguments, whenever they appear in students’ and teachers’ discussions and investigations.

Bourbaki’s play with identities made me think about how easily possibilities for experiencing aesthetic aspects in the classroom, are frozen into pre-determined scripts and pre-determined roles for students and teachers, weaker and stronger students, girls and boys, ethnic groups and teachers.

Reading André Mack’s view on school mathematics, convinced me that educational mathematical practice and artefacts are aesthetic constructs. The questions is not if students should be guided to experience mathematics as wondrous with constructs and ideas, which not only bear meanings but also aesthetic values. The question is how?

Aesthetics’ functional values brought forward by Nathalie Sinclair, made me aware of aspects I had not thought about before. Aesthetic not only can support motivation and generation of new ideas, but also linkages between old and new ideas, and tools for evaluating mathematical models, signs and processes.

Dwelling into the reading of these scholars was the main inspiration for the seven keys model. In the following, I present each key one by one and I present a visual model of the model.

First key is to learn and apply mathematics with the purpose of getting to know and understand relationships in real world contexts. Mathematics is much more than a pure focus on numbers and symbols.

The second key is to compare aspects of mathematical symbols and concepts with the purpose of exploring possible inner harmony or disharmony and possible beauty or ugliness of mathematics.

The third key is to view and use mathematical reasoning as a tool for gaining insight into concepts and activities. Mathematical arguments are not just formal tools for deciding what is true and what is false. They also provide meaning, understanding and images.

The fourth key is to recognize that intuition is crucial, even though it may be difficult to describe what intuition is, and to recognise that intuition is widely applicable, not only in 'emergency' situations where no rules are applicable.

The fifth key is to provide opportunities for sensing mathematical elements, however abstract they may appear. Not only physical representations as centicubes and not only graphical visualisations on paper and screen are to be sensed. All mathematical elements, processes and representations can be sensed as bearing existence and meaningfulness, for instance through visualisations.

The sixth key concerns processual aspects of the learning process. The key is to provide opportunities for participants to experience flow in their learning processes. The key is to avoid falling into traps by mathematical pitfalls, which feels as you were stuck in math holes without any progress.

The seventh key is to accept that mathematics may sometimes be recognisable and familiar, while mathematics at other times is alien and strange. The key is also to accept new light on the familiar, rendering it strange.

The seven keys are visualised as elements on a circle.

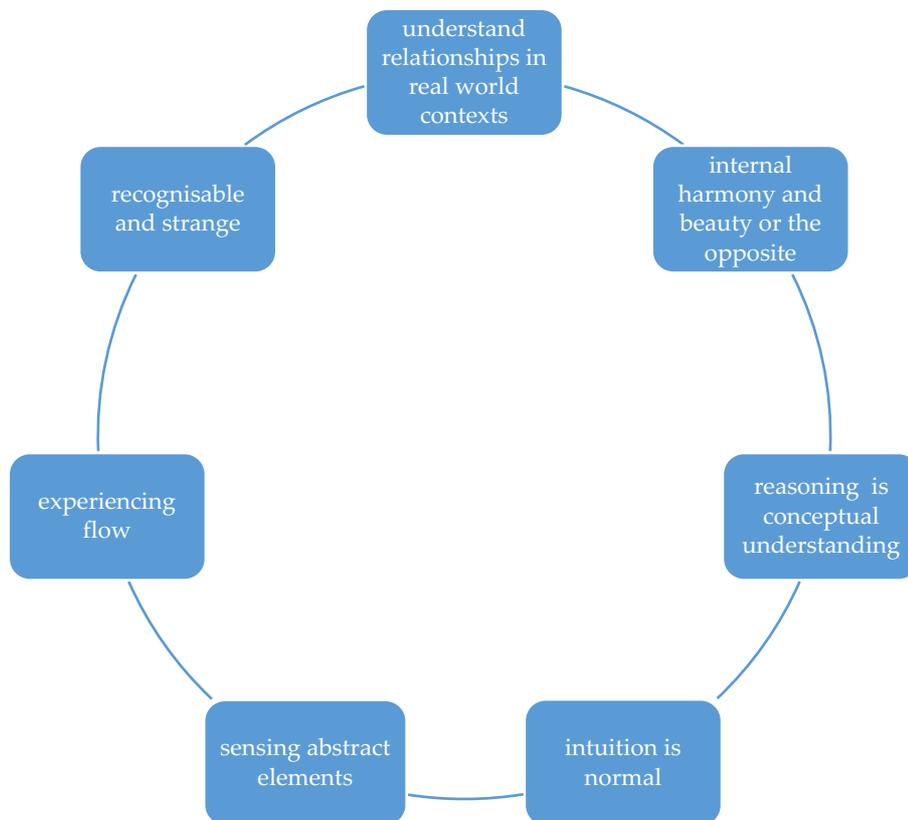


Figure 1. Model of seven keys to open up aesthetic aspects of mathematics in school²

This visualisation shows that the seven keys are equally important, and that the use of the seven keys should not follow a particular order.

National steering documents

Following Mack's recommendation to give students' imagination and fantasy high priority in curricula and standards, I now analyse how students' imagination and creativity are described in Danish national regulations for primary and lower secondary mathematics education (Danish Ministry of Education 2003, 2009, 2017). You may also wonder, why considering aesthetic values in school mathematics, if the national guidelines prioritise very different

² The 7 keys model has not yet been presented in English. Figure 1 is made for MACAS 2017. The seven keys were presented in Danish in Lindenskov, L. (2009). *Æstetiske læreprocesser i matematikundervisning i skolen*. In: Fink-Jensen, K. & Maj Nielsen, A. (Eds.). *Æstetiske læreprocesser: i teori og praksis* (pp. 29-46). Copenhagen: Billesø & Baltzer.

values. As will be shown, aesthetic values are not contradicted by national regulations of the Danish primary and lower secondary school mathematics.

The national general aims for Danish primary and lower secondary school [in Danish: Folkeskolen] include more than knowledge, skills and competences. Also included is imagination (fantasi), which I find important for working with aesthetic aspects of mathematics. Paragraph 2 of the general national aims says

Danish primary and lower secondary school has to develop working methods and create a framework for experience, immersion and enthusiasm, so that students develop knowledge and imagination [in Danish: fantasi] and gain confidence in their own opportunities and background for taking a position and acting. (Law LBK 989, 2017)

Besides national general aims, also national aims for each subject are set up, as well as goals and teaching guidelines for each subject. For the subject mathematics imagination and creativity are part of the aims, but in different wordings before and after 2009.

Before 2009. The aims underlined that the subject mathematics is a creative subject. Individually and in collaboration with others, students should experience that mathematics is a creative subject.

Paragraph 2. Teaching is organized so that students build mathematical knowledge and skills based on their own prerequisites. Self-employed and in community, students must experience that mathematics is both a tool for problem solving and a creative subject in itself. Teaching should enable students to engage and promote their imagination and curiosity.

After 2009. Actually, the subject mathematics is not mentioned. Instead of the subject, the aims put activities in the fore. Individually and in collaboration with others, students should experience that creative activities are required as well as promoted.

Paragraph 2. Teaching is organized so that students independently and through dialogue and collaboration with others can discover that work with mathematics requires and promotes creative activities, and that mathematics provides tools for problem solving, argumentation and communication.

The national teaching guidelines for mathematics (Ministry of Education, 2017b) describe mathematical ways of working. Students' learning should build upon a range of characteristic working habits. As mathematics requires and promotes

creative work, the students should develop their own mathematical results through experimental and examining processes. Teachers must prepare activities that ensure students participate in dialogue and collaboration in order to experience mathematical connections. The students must discuss and test different possibilities, consider alternatives, formulate and test hypotheses, argue for solutions and communicate results.

Therefore, open questions are recommended, to ask for more solutions are recommended, as also to let students investigate if certain connections are possible or not.

Even Grade 1 students should put up questions and hypotheses and investigate in order to support that students develop tools for working in investigative and experimental ways. The basic idea is that this will help also the development of mathematical language and argumentation and mathematical communication during dialogue in smaller groups and during presentations for others.

The concept of creativity is in the guidelines defined this way:

Creativity is the ability to create ideas and to see and create possibilities. Creativity is the ability for problem solving. Innovation is a social proces, where possibilities are identified and where creativity is used to create something new, which is valuable for yourself and for others.

This description of national regulations of the Danish primary and lower secondary school mathematics shows that aesthetic values are in no way contradicted by the regulations. Instead, harmony at the fore. Then a step further through theoretical background, own theoretical model and national guidelines into school practice, seems relevant and obvious to take.

Examples from grade 2 practice in school

The context for the examples comes from an experimental intervention in 2014 in grade 2 mathematics. The intervention design included marginally performing students, meaning low and high performing students. The experiment ran at 41 schools from all over Denmark. In each school, specific mathematics teachers were chosen by the principal to participate. These mathematics teachers where specially trained in a special teacher in service course, directed towards the experiment. The teacher taught 4 students each, 2 low and 2 high performers. The teaching was one-to-one teaching in 48 sessions for a period of 12 weeks. A total

of 281 intervention students participated (Lindenskov, 2013), (Metropolitan University College & Aarhus University, 2016). The experiment is studied in a RCT-study. The experiment is also studied in other kinds of study, mainly from intervention programme theory/realistic evaluation to investigate what works for whom, in what circumstances, in what respects, and how (Pawson et al, 2004). Preliminary findings from analyses of video-recorded instruction sessions suggest that an intense (Massumi, 2002) mathematics dialogue between teacher and student is a decisive programme mechanism for positive effect (Lindenskov & Kirsted, 2017). It is in intense mathematics dialogues we find the seven keys model exemplified from the video-study. All the provided examples are from the same low performing student.

First key is to learn and apply mathematics with the purpose of getting to know and understand relationships in real world contexts. The example shows the teacher encourages the student to listen to, react to and construct stories about phenomena from the student's life, which the student likes. They are talking about some favourite cakes. Each time the meaning of zero turns up in the stories, the student quickly shifts from one chair to another.

Student: Yes, I do like cakes, but not birthday cakes. I like dream cake [Danish: drømmekage].

Teacher: I once made a dream cake....They then had nothing left.....nought.....zero...nothing at all.....none.....my efforts came to naught.....empty..... half empty..... on an empty stomachreturn empty-handed... Then the cake is ready.

Student: And then you have to wait until the guests arrive.

Second key is to compare aspects of mathematical symbols and concepts with the purpose of exploring possible inner harmony or disharmony and possible beauty or ugliness of mathematics. First, the teacher tells another zero-story about buying and bringing back trousers from a local store.

Student: Yes, I lend 300 kroner in the bank and pay them back. Then I have no money.

Teacher: Shows a brick with ten written on it, then twenty.

Student: Eij, it is funny, I see one and two.

Teacher: Here some more (tens), do they have anything in common?

Student: No, they differ, look! (points at 1,2,3....)

Teacher: Please find all the zeroes (and the student does)

Third key is to view and use mathematical reasoning as a tool for gaining insight into concepts and activities. Mathematical arguments provide meaning, understanding and images. The teacher wish to discuss arithmetical expressions.

Teacher: If you have a task $5 - 5$

Student: Yes, it gives 5.

Teacher: Look again

Student: It gives zero

Teacher: What about $4 + 0$

Student: 4

Teacher; Oh, but why?

Student: Because zero is nothing

Fourth key is to recognize that intuition is crucial and widely applicable. Fifth key is that students sense mathematical elements with meaning. Teacher formulates an open situation, where geometric forms and patterns can occur. Teacher lets the student do things himself and raise questions, which the teacher reacts to in non-scholastic ways.

Teacher: Which kind of patterns do you see in a star?

Student: It is coloured, right?

Teacher: It may be coloured. Can you draw a star?

Student: No.

Teacher: Okay. When I myself draw a star, I draw a triangle and a triangle. Like this. Do you see any patterns now?

Students: Can I try?

Teacher: Sure.

Students: How reard. [In Danish: Hvor er det sygt.] Is it okay, that I look at your star?

Sixth key is experiencing flow in the learning processes, the opposite of being trapped without any progress. The teacher and the student are working on wooden bricks with different geometrical forms.

- Teacher: Maybe we can use that red one. Do you see what we get now?
- Student: A pizza.
- Teacher: Yes! A pizza, carved in some pieces: What is the name of that form?
- Student: Eij, now I know, what to do.
- Teacher: What could you do?
- Student: Now it is with pepperoni on.....tomatoes, cheese, and topped with nachos.
- Teacher: Nachos, yes. Okay. Okay, now I just [Danish: lige] ask you, which forms do you have?
- Student: Triangles.... Hexagons.... and a big cheese..... and ham

Seventh key is to accept that mathematics may be familiar, as well as alien. This video analysis did not exemplify this key.

Final remarks

National regulations of the Danish primary and lower secondary school mathematics do not prohibit that aesthetic values are being prioritised in practice. Using the seven keys model gave some optimism to let further emphasis be put on aesthetic values in mathematics classrooms. Nevertheless, the seven keys model has a prerequisite, that the teachers have time and competences to listen to and communicate with the students in intense mathematics dialogues.

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SHOULD MATHEMATICS BE A CREATIVE SUBJECT? HOW IS THIS REALIZED IN PRACTICE IN DENMARK?

Lisser Rye Ejersbo¹

Abstract

The ministerial objectives for mathematics education in the Danish Folkeskole (grades K-ten) state that students will learn that mathematics is both a tool for problem-solving and a creative subject. In this article, I explore how teachers in Denmark meet the challenge of teaching mathematics as a creative subject. How do they understand creativity and how do they realize it in practice? And what makes mathematics a creative subject? Is it the task, the way it is performed or the relationship between teacher and students? The crucial difference between different classrooms seems to be the ways in which teachers prepare and present mathematical themes. This seems to depend on how well they know their students and how they understand learning processes. In this article, I present three cases: The first one takes place in second grade, the second in ninth grade and the third in first grade. The discussion will focus on how to make mathematics a creative subject.

Introduction

The ministerial objectives for mathematics education in the Danish Folkeskole (grades K-ten) are formulated in a list of 'what', 'how' and 'why'. Under 'how', it is stated that:

The teaching shall be organized so that the pupils build up mathematical knowledge and proficiency on the basis of their prerequisites. The pupils shall, independently and together, learn that mathematics is both a tool for problem-solving and a creative subject. The teaching must give the pupils a sympathetic insight and further their imagination and curiosity. (Official translation from uvm.dk, retrieved 2017, February 17)

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This article explores how Danish teachers meet the challenge of teaching mathematics as a creative subject. How do they understand creativity and how do they realize it in practice?

In the article, I present three cases:

- The first case takes place in a second grade classroom, where a teacher plans to introduce a theme about single digit addition.
- The second case takes place in ninth grade, where the students work with open and self-designed tasks.
- The third case takes place in a first grade classroom, where the teacher uses small plastic bears with different personalities to encourage students to solve problems.

All data are collected using video or audio recordings, or via observation notes. Teaching in mathematics has to be creative, but what is creativity? Lene Tanggaard, Aalborg University, writes (2016):

“Creativity is bounded to the context and deals with the ability to create new solutions in this context. Creativity build directly or indirectly on knowledge which is available.” (LRE translation)

We can also regard our brains as a prediction machine (Clark, 2013). We use our memory to predict new situation. In this perspective, creativity is to imagine what is not yet there. To place ourselves in new perspectives and imagine how the world looks from another person’s perspective. In the cases presented here, the question I ask is how the teacher uses his or her creativity to emphasize the students’ creativity in solving mathematical problems.

The first case

We are in second grade classroom at the beginning of the school year. The class contains nine boys and twelve girls. Eleven of the students are from ethnic minority backgrounds and do not fully master the Danish language. There are two teachers in the classroom and I follow one of them, who teaches only five of the students.

In the textbook, there are tasks concerning how to add 2 and 4, illustrated, respectively, using the legs of ducks and sheep.

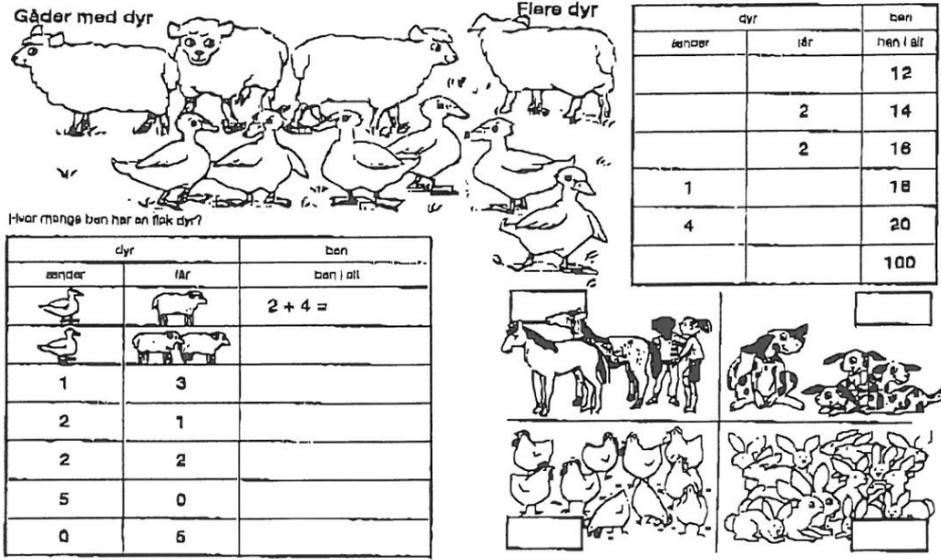
Gåder med dyr

Flere dyr

Hvor mange ben har en flok dyr?

dyr		ben
ænder	lår	hen i alt
		12
	2	14
	2	16
1		18
4		20
		100

dyr		ben
ænder	lår	ben i alt
		2 + 4 =
1	3	
2	1	
2	2	
5	0	
0	5	



Figur 1

Figure 1: A page from the textbook. The students have to fill in the empty spaces.

'Gåder' means riddles.

The class has also been working with animals on a farm and have visited a farm. Therefore, the teacher expects the students to recognize the animals depicted on the page in the textbook.

The teacher has prepared by asking the students to quickly draw one of each animal, a duck and a sheep, before doing the tasks from the textbook. Once the students have finished these drawings, the teacher collects all five of them with the purpose of looking at them together with the students. The drawings surprise the teachers as in each case, students have drawn the two animals with an identical number of legs:



Figure 2: Two of the drawings, showing that the students draw either four legs or two legs on both animals. 'Elev' means student.

When the teacher planned this lesson, she was convinced that the children knew that a duck has two legs and a sheep has four. However, this knowledge is not reflected in the students' drawings. Why is that? Is how many legs the different animals have unimportant to the students or do they simply not know? Perhaps it is a bit too abstract for the students to think about animals' legs? The students

do not know why they have to draw these two animals, which is why we see the intuitive perception of these animals. These thoughts are the background for the following exchange between the teacher and the students:

The teacher: How can we see the difference between the sheep and the duck?

A student: The sheep has curls, which the duck doesn't have.

The teacher: What about their legs? Do they have the same number of legs?

A student: Oh no...

The students mention curls as the significant difference. Only when the teacher asks about the legs do the students react. After this exchange, the students quickly draw the animals again.

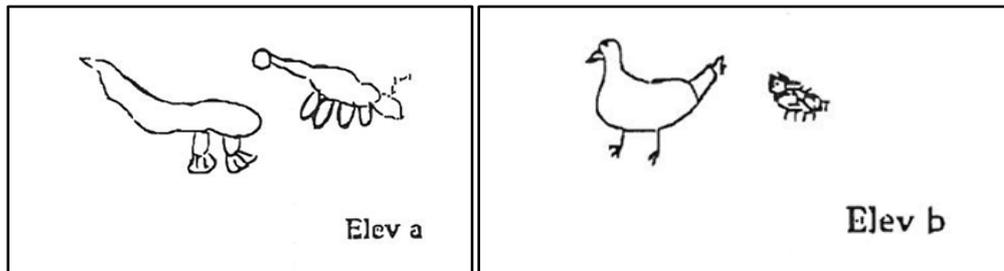


Figure 3: New drawings from the same students. Here, they differentiate between the number of legs on the sheep and the duck. 'Elev' means student.

The rest of the lesson is spent solving the tasks from the textbook. After a short time, one of the students makes a symbol for four (the sheep) and a symbol for two (the duck). Another chooses to use small cubes as help. Now things move fast and all the students are engaged in solving the tasks.

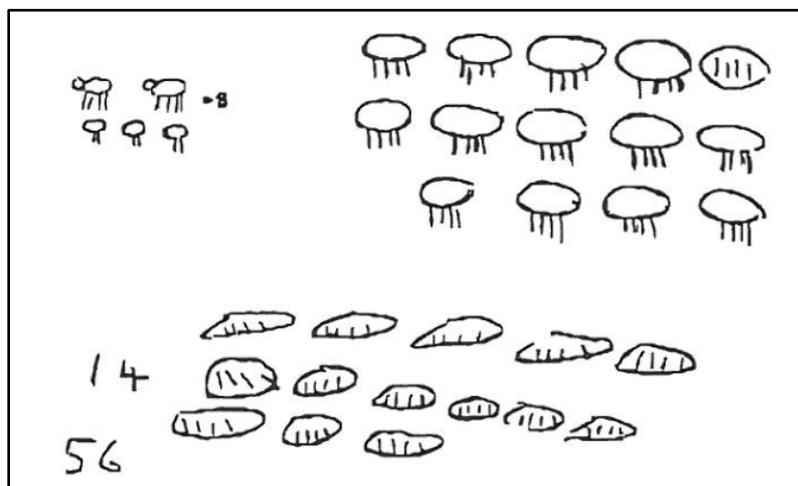


Figure 4: The drawings the students use to solve the tasks.

The two teachers prepared – in a creative way – to start the lesson with a drawing. They imagined that the drawings could be a good way of initiating a dialogue about how the students might solve the tasks. Little did they know that the students had their own ideas about what was significant for each of these animals. Funnily enough, all five students, which I followed, did the same thing. Nevertheless, the lesson was actually a success in terms of getting the students to create and work with concepts of the numbers two and four. By asking the students about their drawings – and, parallel with this, their understandings – it was possible for the teacher, who I followed, to help the students taking into account their own perspectives, rather than based on assumptions. The drawings gave the teacher the necessary information about where it was appropriate to start. In this way, the students identified the focus for the rest of the lesson.

Normally the teacher prepares and plans how to reach the lesson's learning objectives, but there are many ways to do that. Starting with the students' thoughts is a way of including informal knowledge in the learning space created by the teacher. Even though the students draw the animals incorrectly, she does not correct them; instead, she asks them 'How can we see the difference?' and 'Do they have the same number of legs?' This is one of the ways in which the teacher in this case shows her creativity. She planned a creative start, which did not go as predicted, but, using her experience and creativity, she asked the students questions that helped them develop their own understanding of the mathematical concept of the numbers two and four. Their understandings thereby developed from the idiosyncratic to embrace the mathematical concept of the two numbers.

The second case

We are in a ninth grade classroom during the spring. The students have an upcoming oral examination in mathematics in front of them and the teacher has prepared to work with open tasks in order to train oral communication.

The teacher introduces the tasks and methods for the day in the same way the oral examination would take place, he tells them. The sheets with the day's tasks contain a lot of information about football and, based on this information, the students are expected to develop their own tasks. This information is accompanied by a list of ideas, such as: calculate the distance, statistical calculations, scale of the football ground, area and volume of a football, etc. Groups of three are established and the students read the exposition and try to formulate new tasks. The teacher walks around in the classroom and

helps the students as needed.

A student: But there are no questions, are we just making some on our own?

This is precisely what is intended, but the students do not receive any guidelines or advice regarding how to formulate such questions. Most groups produce graphical representations of various tables; e.g. a pie chart, a curve diagram or bar chart. One of the groups chooses to work with something else, deciding to calculate the volume of a football net, which is illustrated in the sheets they are given:

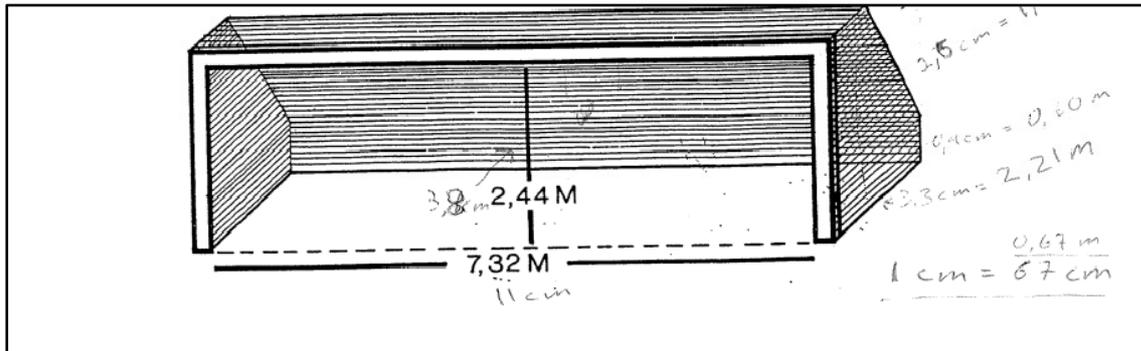


Figure 5: A copy of the football task

This particular group consists of three pupils: one girl and two boys (whom I will call Line, Carl and Emil). They are all interested in football and enjoy mathematics. The questions they make up are based on their interest in football, so even though they are not especially interested in the answer to their own questions, they enjoy the process, finding it fun. Nevertheless, their questions are both strange and difficult.

[149] EMIL: And then it is difficult, I don't know how to explain it

[150] CARL: Yes, it is difficult to explain and it is not correct either, because we don't have the right measures, but anyhow...

All three of them continue in an enthusiastic way with the work. The teacher approaches the group:

[163] Teacher: Can you calculate anything from these measurements here?

[164] CARL: We know that this (points at a line on his paper) ... and we know that this...

[165] LINE: We found the area and the circumference of this (points at (?))

- [166] EMIL: Yes, we found the area and the volume.
- [167] CARL: What do you mean by that? (Addressing the teacher)
- [168] JOHN: The measurements on the drawing, what are they in reality?
- [169] CARL: The measurements on the drawing?
- [173] LINE: Then we need to find the scale. That's the reason why we couldn't...
- [176] CARL: But we don't know how much these measurements are. If we have 32 cm this way then... (...) What about the funny side, we don't get any information about that (points at the sloping back side)

This kind of discussion goes on for several minutes and the teacher leaves the group without giving them any hints regarding what they could do next. They tried to calculate the volume of a football net drawn in perspective. The teacher helped them to find a scale they could use to find the real measurements and how they could find volumes of different prisms, but he never asked them who might be interested in knowing the results; there was a total lack of authenticity. They had difficulties finding the right scale because the perspective deceives them [176]. However, they accepted some mistakes when they found things too difficult; they knew what they had written was wrong but were not bothered because they did not need to use the result for anything. The teacher's mathematical communication with the pupils was influenced by his beliefs. He did not ask precise questions, but instead commented on some of the things the pupils said or did. This strategy balances on the edge of 'anything goes'. He only asked questions about mathematics; he never asked what the results could be used for. At the same time, the students' motivation derived from the football theme, and they wanted to combine difficulty with something related to football: "It was difficult and tough". The result was not of interest to anybody, and the answer they found was not correct either, but they did not care; and the teacher did not care either. Was this a creative mathematical problem? The students were preoccupied with doing mathematics, but the result was unimportant. The process was only meaningful to them because it concerned football. And, because they were good at mathematics, they wanted to work 'with something difficult'. However, they were not given any tools for finding out how to measure a length drawn in perspective. The calculation was numerical, but they did not care that they 'cheated' to get the result. They were not really challenged, as they could

have been had they been asked to work with how perspective drawings can, in fact, be measured and to explain who could benefit from the results.

The evaluation the teacher himself conducted of the lesson showed that all the students expressed satisfaction and that they had enjoyed the lesson; they felt they had learned a lot and were allowed to make their own choices, they liked that they were encouraged to work together and they all expressed that they had fun. Not one of the students was dissatisfied.

The teacher himself was satisfied with the lesson and the outcome, because the students seemed satisfied and felt that they had learned some mathematics. The teacher's creativity consisted of the open tasks and the organization of the work. The goal for the lesson was not clear: as long as the students did something mathematics-related, it seemed as though anything goes. Therefore, the goal became that the students were satisfied and felt that they were learning mathematics. The reason for using a real-world and open-ended problem was to help make mathematics meaningful to the students, but very little attention was paid to the authenticity of the real-world problems.

The third case

We are in a first grade classroom during the spring with 21 students: 10 girls and 11 boys, 15 of whom are from ethnic minority backgrounds. In the case, we look at an innovative use of hands-on material for students in the early grades: first, second and third. The particular hands-on material consists of various small plastic bears in three different sizes and weights and in four different colors – blue, green, yellow and red. These bears are used quite frequently in the mathematics lessons in Denmark. In the teacher's guide provided by the manufacturer, the bears are recommended for use as a means of cognitive help with sorting, pattern recognition, counting, early arithmetic and exercises with weighting (Alinea, 2015).



Figure 6: The twelve bears in different sizes and colors

The mathematics teacher normally uses a wide variety of hands-on materials. Many years ago, he developed an accompanying fairytale for the bears, because he saw the possibilities for these bears to be displayed as actors in a play. He created a world where the group of 12 different bears could have many experiences. In the world of the bears, relatively normal things exist, such as a circus, school, fishing rods, hot air balloons, etc. - things that the students are familiar with, or that they could at least easily imagine. Some of the bears like competitions and they frequently run into all kinds of trouble, so the students need to help them by solving some calculation problems. The teacher also gave the 12 bears names and different personalities in the play. The names are close to the color and size in the sense that the first letter in the color is also the first letter in the name; for instance, the biggest blue bear is called Badut and the little blue bear is called Benjamin. The bear Badut plays a special role. Its personality is a combination of both 'good' and 'bad' characteristics and he likes to test the limits and venture into prohibited zones. He likes to brag and compete, especially with the big red bear called Rumsak, which is very strong but not as smart as Badut. Badut also protects Benjamin, who is a small and timid bear, but is very nice and has a good sense of humor. Most of the bears have such dual characteristics. The play was developed about eight years ago with a first grade class, but the figures are still alive and are used in the daily math teaching.

In today's lesson, the bears are invited to the Arithmetic University, which is part of the fantasy bear-world where both Assistant Professor SmartCount (short for smart counting) and Professor CalStrat (short for calculation strategy) work. It is SmartCount's birthday today, and he is invited to a birthday party, where the bears play a ritual game with chairs. Each bear has a number, and now SmartCount chooses three numbers to stand on three chairs in a row. The goal is that the three numbers should be added in an easy way. The first three numbers chosen for the game are 6, 9 and 1. This means that the three bears with these numbers have to climb up on the chairs, but the order matters. In what order should they stand to be added in the easiest way?

The students are very engaged in this story and many students raise their hand to suggest an order and an explanation for why their suggestion is a good idea. At this point, the teacher says: Badut has number 6 and he wants to stand on the first chair. What do you think of that?

The students have a few minutes to discuss in small groups whether Badut's wish is a good idea or not, and why. Badut is a very special bear and the students have mixed feelings about him, but he always makes them laugh. Now he wants to be the first. After a while, when the students resume the plenary discussion, one girl suggests that Badut should know that it is also nice to be the last one, because it is too difficult to start with the 6; it is much easier to start with 9 followed by 1.

After this introduction, the students work with the order of other numbers given by the teacher as the row from SmartCount. The students can use the bears to arrange different sequences and write the results on a piece of paper. A picture of all the bears with their different numbers is displayed on the whiteboard. The students laugh and discuss in small groups how they should place the bears. At the end, the whole group discusses the results and the argumentation for the preferred choices.

Later on in the same lesson, the students play a game, where a table is used to represent a classroom containing tables and chairs for all the bears. The bears are standing outside the classroom-table and our task is to get them into the classroom. I play with Amir, one of the students in the first grade. We have a dice with both positive and negative numbers and take turns to throw it. We both add and subtract. The winner is the one who gets the last bear into the classroom. The game is easy for Amir; he counts or just moves the bears where they should go. Amir has played this game before and helps me when necessary. He is also the winner of this game, but after having placed all the bears inside the classroom, he takes the bear Badut outside the classroom again. I ask him:

- I: Why are you doing that; you just won because you had them all inside the classroom?
- A: Yes, but Badut doesn't like to stay in the classroom for long.
- I: What happens with him if he stays in the classroom for too long?
- A: He gets a little crazy, so it is better that he just leaves the classroom (a little break) sometimes.

The learning goal for this mathematics class is to practice addition and subtraction of small numbers; yet, using the bear game, it becomes much more. The first task could be viewed as an open task, where the students look for good "number pairs" or other patterns and explain why this is the easiest way to add the three numbers. In the case described here, the challenge is to find why it is not such a good idea that Badut, with the number 6, should be the first number to add, and the students are engaged. The world of the bear-group functions as a fairytale for the students. Bettelheim (1976) describes how fairytales evoke meaning and awake children's curiosity, which is clearly observable in this setting, and how it helps children to accept themselves and others. The students like to go into the stories of the bears and it seems easier for them to believe that they can solve and discuss the mathematical problems when they do it for the bears, rather than solving tasks from a textbook for themselves.

As for the second activity, there is no doubt that Amir is talking about himself when he lifts Badut out of the classroom, but at the same time, he is able to stay in the classroom, seemingly without problems. Amir's identification with Badut seems to release him from such pressure and, at the same time, he is working with mathematics. Working with the bears and their different personalities enables the students to identify, like and tolerate different personalities. Of course, they know that they are only toy bears; nevertheless, when they work with their problems, they stay in the magic circle that the story creates (Steen, 2001).

The students attend a normal Danish elementary school placed in a district with many different nationalities, all speaking Danish at school, but many different languages at home. Statistically, foreign students get lower marks on the national tests (Christensen, Egelund, Fredslund, Jensen, 2012). Yet the students in this particular class, who learn mathematics through acting with the bears, among other things, score significantly above the Danish average. Students with ethnic minority backgrounds are not treated differently in this class; all the students are treated in the same way and work with the same materials. Using the hands-on material in this particular way seems to benefit all students.

The creativity lies in both the preparation and in the realization of the lesson. The force of the narrative used as a kind of fairytale helps the students in their arithmetical learning processes. It injects meaning into the learning process and helps the students to concentrate on solving problems in mathematics. The challenge is how to create these stories in ways that engage the students. Doing so demands a mixture of humor and creativity, combined with a sense of how to make solving mathematical problems a natural and meaningful activity. Dietinger (2015) discusses the power of the metaphor of mathematics curriculum as story. He shows how the mathematical characters and their actions create this power.

Conclusion

The three cases are examples of different Danish teachers' contributions to how creativity can be used in teaching mathematics. None of the cases focuses on mathematics itself as a creative topic, but rather on how teaching mathematics can be creative in different ways. Teaching mostly consists of communication between the teachers and students, among the students or with the textbook. In the three cases, I have focused mostly on how the teacher communicated and how the students worked with mathematics as a reaction to this communication. Yackel & Cobb (1996) describe through examples how important teachers' questions are for what happens in the classroom, and we know from many other examples in the area of math didactics that tasks and questions are crucial. With

inspiration from Stieg Mellin-Olsen (SMO), Skemp (1976) describes two different ways to understand the word 'understanding' when teaching mathematics. SMO distinguished between the two by calling them 'relational understanding' and 'instrumental understanding'. Relational understanding means knowing both what to do and why, while instrumental understanding means an understanding of knowing what to do to get the right answer for a given task. In the first and third case, the teachers used their creativity to help the students to understand and be able to solve problems. In the second case, the teacher wanted to help the students understand how mathematics could be used to solve their own problems. This could give an insight into how to solve problems, which is closer to instrumental understanding – but without a focus on the outcome. There is no recipe for creativity when teaching mathematics; rather, it is the ability to create new solutions in a familiar context, where creativity builds directly or indirectly on available knowledge. To be creative is to imagine what is not yet there; to imagine what could happen and how students will react.

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CONTRIBUTING TO STUDENTS' PERCEPTION OF THE RELEVANCE AND APPLICATION OF MATHEMATICS BY FOCUSING ON THEIR MATHEMATICS-RELATED BELIEFS

Maria Kirstine Østergaard¹

Abstract

There seems to be a connection between students' general perception of mathematics as irrelevant and isolated from other subjects and a schema or algorithm-oriented approach to mathematics. Such an approach to mathematics is related to negative beliefs about mathematics (Grigutsch, 1998), and combined with the apprehension that good mathematical skills are a sign of intelligence, the schema-oriented approach can even be connected to low self-efficacy or self-esteem in mathematics (Malmivuori, 1996). By combining the Autonomous Learning Behavior Model (Fennema, 1989) and the Cycle of Mathematics Avoidance (Preis & Biggs, 2001) in the Model of Self-Reinforcing Effect of Negative Mathematics-Related Beliefs, I present how negative beliefs about mathematics can be self-reinforcing. I therefore argue that it is essential to focus on the development of students' beliefs in mathematics education, particularly about mathematics as a discipline, in order to enhance the students' apprehension of the role and use of mathematics in the world and to emphasize the interdisciplinary possibilities of mathematics. Working with mathematics-related beliefs as a goal can provide the relevance and motivation for students to change their negative beliefs about mathematics as well as broaden their perspective in other subjects

Introduction

Whereas mathematicians commonly perceive mathematics as aesthetic, creative and challenging (Devlin, 1994), many students – and people in general – view mathematics as a tool for calculations or as a set of rules (Boaler, 2016). Grigutsch (1998) calls this rule based view on mathematics a 'schema-oriented' approach. It involves perceiving mathematics as a set of formulas or procedures to be

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followed in order to reach the right result, or even as a set of rules to be memorized just to be able to pass a test. It may also include believing that there is only one correct answer to a mathematical problem. Both Grigutsch and Malmivuori (1996) present studies that show how this approach is very often connected to a low self-image as a mathematics learner and to negative beliefs about mathematics. Malmivuori also finds that if these beliefs are combined with the belief that a special talent is needed to excel in mathematics, it has an even larger impact on the mathematical self-esteem. Contrary to this, a more process-oriented approach and beliefs that mathematics is possible to learn for everybody are related to a more positive self-image (Grigutsch, 1998).

Beliefs about mathematics

To investigate how beliefs about mathematics might be defining for the way we feel about the subject, we need to take a closer look at the concept of mathematics-related beliefs. Philipp (2007, p. 258) defines beliefs as “lenses through which one looks when interpreting the world”. Op't Eynde et al. (2002) categorizes a belief system in three clusters or dimensions: beliefs about the object, the social context and the self. In relation to students' mathematics-related beliefs, the object is mathematics education, the social context is the mathematics classroom, and the self refers to the self-perception in mathematical situations. Jankvist (2015) expands this framework by adding the dimension of 'mathematics as a discipline'. This fourth dimension exceeds the content of 'mathematics as a subject' in the original model, which only concerns educational perspectives of mathematics. 'Mathematics as a discipline' is about the role of mathematics in the world.

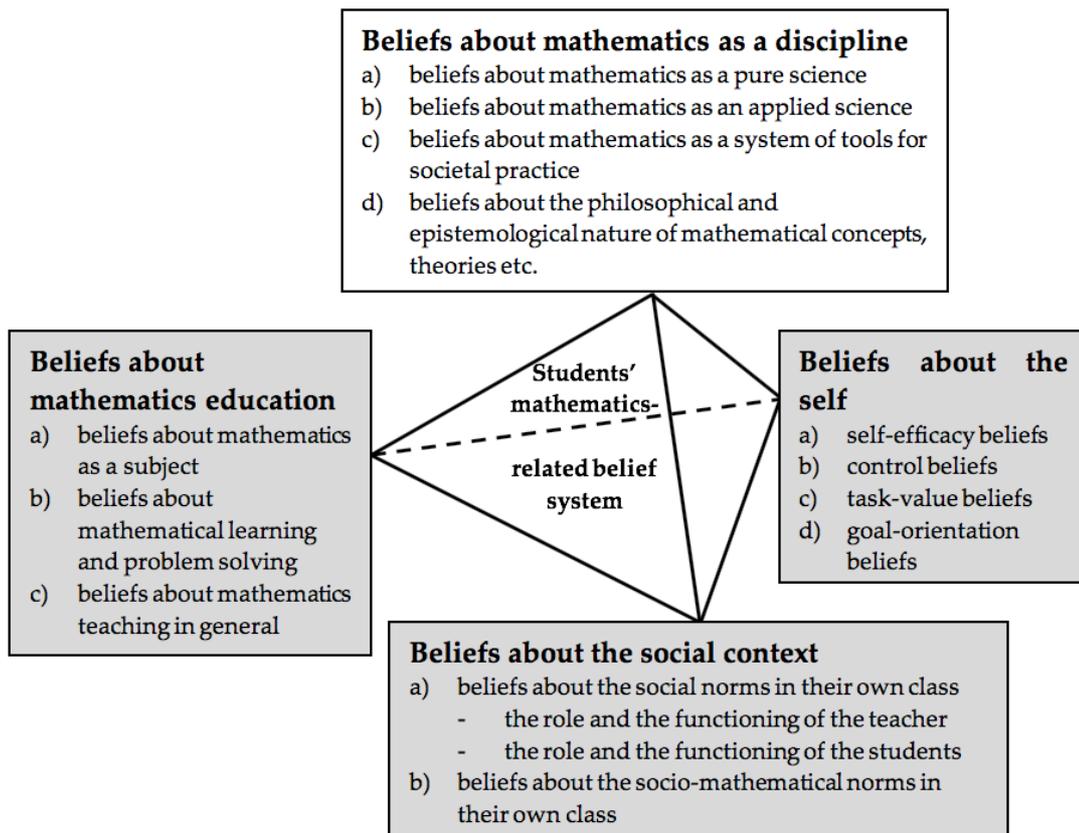


Fig. 1: Students' mathematics-related belief system (Jankvist, 2015). Original model (grey boxes) from Op't Eynde, de Corte, and Verschaffel (2002)

When mathematics is taught in a certain way, it affects all four dimensions of the belief system. Not only does it form the way we think about mathematics as a subject, about what a mathematics lesson consists of, how one solves a mathematical problem or how mathematics is taught; it also contributes to our perception about the roles of teacher and students in the class and the way we talk about mathematics, as well as our self-image as mathematics learners and our learning behaviour. Finally, it affects our basic idea about the nature of mathematics, what it is used for and what part it plays in society. So, if mathematics education is based on a schema-oriented approach, it will influence the students' mathematics-related belief system as a whole.

As earlier stated, the schema-oriented approach is related to students' low self-image. But it also seems to be related to their perception of the relevance of mathematics. Believing that mathematics is a set of rules or only focused on finding the right result, might increase the possibility that one does not see a connection between mathematics and the real world. Thus, mathematics appears irrelevant to everyday life and isolated from other subjects. Mathematics has a life on its own, so to speak.

In the summer of 2016, I interviewed four adults suffering from math anxiety as part of a study aiming to examine whether correlations between math anxiety and beliefs about mathematics can be observed. Hence, the interviews focused on the participants' beliefs about mathematics. Some of their statements clearly depict the above-mentioned connection between the schema-oriented approach to mathematics, the belief that mathematical talent is innate and negative beliefs about mathematics (including low self-image):

"In school, it was all about finding the right result. You had to perform all the time."

"You had to find the right result, and I couldn't do that, because I was thinking wrong."

"Math is just not for me. I simply don't have the talent."

"There is no purpose with math."

"For me, it's just memorization."

"I never meet mathematics. Only in a school context."

These quotes show how people with a very strained relationship to mathematics experience a distance between themselves and mathematics and between the real world and mathematics. Negative beliefs like these might originate from negative experiences (Chaman, Beswick, & Callingham, 2014) or from external influences (parents, teachers, media etc.) (Gunderson, Ramirez, Levine, & Beilock, 2012; Lake & Kelly, 2014).

The self-reinforcing effect of negative beliefs

Associating mathematics with something unpleasant is often the consequence of having **negative experiences** with mathematics. A typical reaction is avoiding what makes you feel anxious. Avoidance is a common defence mechanism in relation to mathematical difficulties in general and to mathematics anxiety in particular. To show how negative experiences might lead to avoidance and poor mathematics performance, which again will cause negative experiences, Preis and Biggs (2001) set up The Cycle of Mathematics Avoidance (fig. 2):

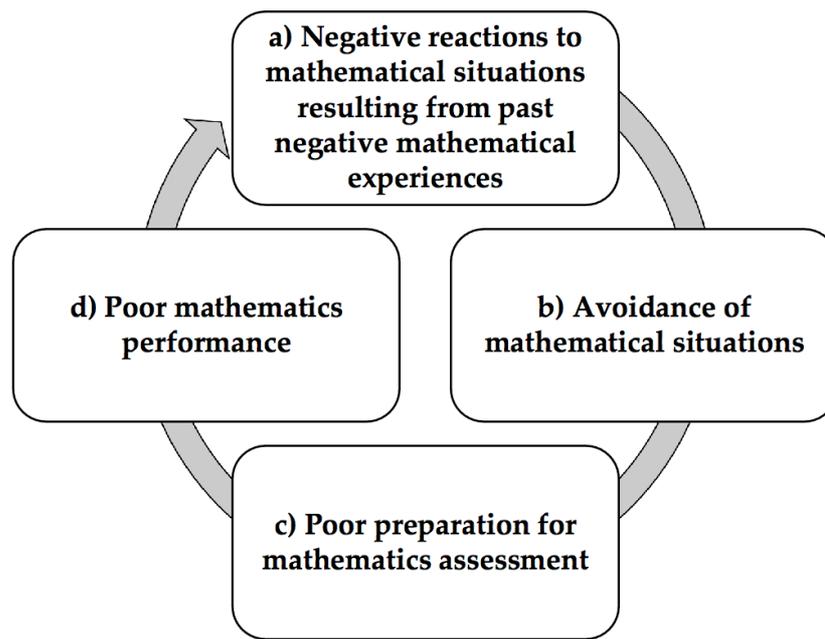


Fig. 2: The Cycle of Mathematics Avoidance (Preis & Biggs, 2001, my illustration)

Avoidance of mathematics can be perceived as a change in learning behaviour. According to Fennema's Autonomous Learning Behaviour Model (fig. 3), two factors affect students' learning behaviour: the internal belief system and external or societal influences.

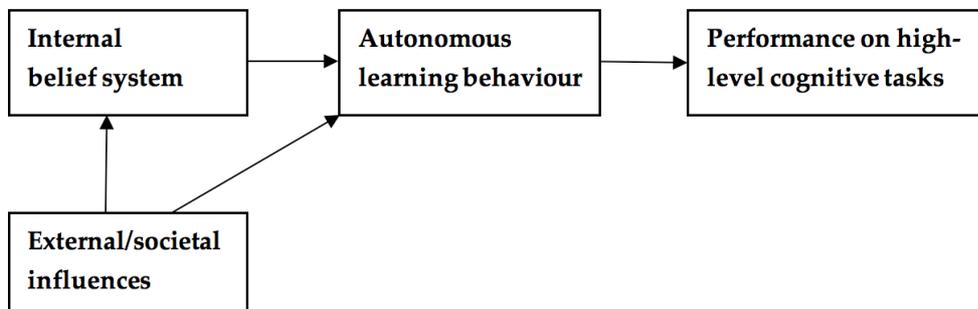


Fig. 3: The Autonomous Learning Behaviour Model (Fennema, 1989, p. 213).

This means that negative external influences, i.e. from school, peers, parents, media etc., can create negative beliefs about mathematics *and* lead to a depressive learning behaviour where the student is less prone to paying attention, participating in class or doing homework (Ashcraft, Kirk, & Hopko, 1998). Such a learning behaviour will inhibit the development of mathematical competencies and thereby the mathematical performance.

Combining these two models illustrates how external influences (including negative experiences) affects both the beliefs about mathematics, the beliefs about the self and the learning behaviour. The learning behaviour is defining for the achieved mathematical competencies which again is reflected in the mathematics

performance. A poor performance means more negative experiences and more negative reactions. This again will influence the beliefs about mathematics and about the self. Thus, negative beliefs about mathematics seem to be self-reinforcing (fig. 4).

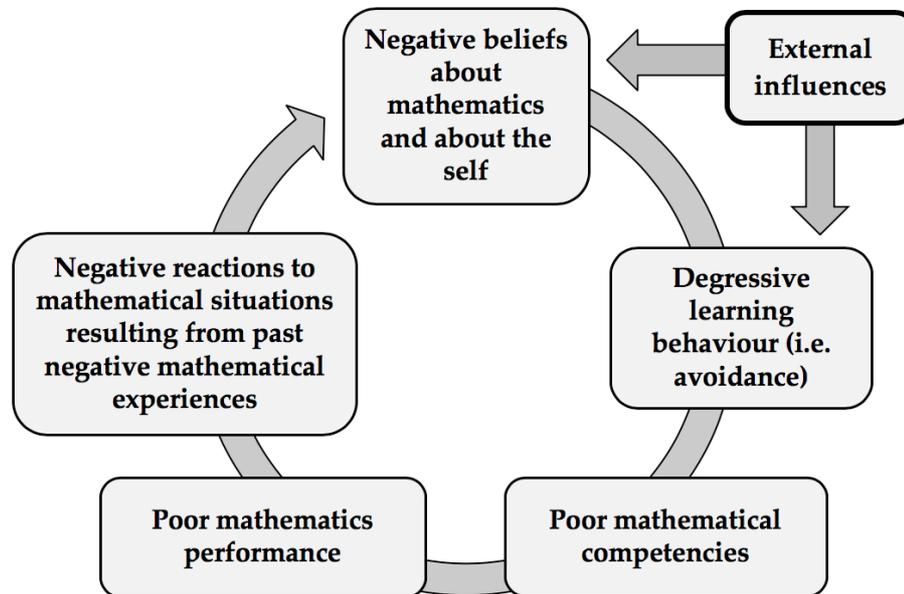


Fig. 4: Self-reinforcing effect of negative mathematics-related beliefs (my illustration and combination of 'Cycle of Mathematics Avoidance' (Preis & Biggs, 2001) and 'Autonomous Learning Behaviour' (Fennema, 1989))

Thereby the external influences become central to the character of the mathematics-related beliefs. If these external influences are negative – parents who tell stories about their own struggles with mathematics, a teacher who has a low tolerance for mistakes, classmates who laugh at an incorrect answer, cartoons that stereotype mathematicians as boring or only male, experiences of failure in mathematics lessons, fights with parents over mathematics homework etc. – the beliefs about mathematics and about oneself as a mathematics learner are likely to become negative as well. This will influence the learning behaviour: perhaps by being more passive during mathematics lessons, skipping homework, giving up more easily when working on mathematical problems or refusing help. In any case, it will most likely have a negative effect on the learning of mathematics and on the mathematical performance. This again causes more negative experiences with mathematics and can eventually initiate the development of protecting beliefs (Goldin, Rösken, & Törner, 2009) such as convincing oneself that mathematics is irrelevant, that it is only needed in school or that it is impossible to learn without a special talent. The self-reinforcing effect of negative beliefs may even have psychological consequences and lead to mathematics anxiety.

Thus, the model indicates that one of the only ways to stop, reverse or even prevent this self-reinforcing effect of negative beliefs is by changing the external influences.

Mathematics education as an external influence

The school's mathematics education accounts for a large part of the external influences. This is the context in which most people have the majority of their mathematical experiences. This means that mathematics educators play an important role in the development of students' mathematics-related belief system.

Consequently, a teaching structure which presents mathematics as rule-based and schema-oriented and which takes place in a task and performance culture, potentially has a negative effect on the students' learning behaviour and their mathematics-related beliefs. Signaling that mathematics is mainly about being able to quickly find the correct solution to mathematical problems, will enhance the possibility that students who work at a slower pace, value understanding over result, or consider the context of the problem, will fail or feel that they cannot live up to the expected standards. The risk of having negative experiences is therefore higher for these students. As a defence mechanism, these students are likely to start avoiding mathematics (Chaman et al., 2014). If students moreover believe that a special talent is needed to succeed in math, they are even more likely to reject it ("mathematics is just not for me and I will never be able to learn") (Goldin et al., 2009). These negative beliefs are often easier to live with if the subject you are rejecting is not important. Therefore, many students who experience difficulties with mathematics will downplay the relevance of mathematics and its role in everyday life and in society, as was the case with the four people I interviewed.

Compared to fig. 4, it means that a schema-oriented teaching structure increases the risk of developing negative beliefs, which then can be self-reinforcing. If a student has developed beliefs about mathematics as irrelevant and impossible to learn, it is very likely that the learning behaviour will degress even more, which will lead to further negative experiences.

Focusing on 'mathematics as a discipline'

If we take a closer look at the belief dimension which concerns mathematics as a discipline, we find that it contains elements that offer a *context* for mathematics:

Beliefs about mathematics as a discipline

- a) beliefs about mathematics as a pure science
- b) beliefs about mathematics as an applied science
- c) beliefs about mathematics as a system of tools for societal practice
- d) beliefs about the philosophical and epistemological nature of mathematical concepts, theories etc.

'Mathematics as a discipline' includes beliefs about the nature and the application of mathematics and can provide a connection to other subjects and to the real world, which is precisely what the people I interviewed had trouble recognizing! They all expressed that the lack of relevance has been a major inhibitory factor in their mathematics learning, and they all found it difficult to explain the application of the mathematics taught in school. Thus, if the mathematics education is to contribute to a change in the way the students view mathematics, it seems very appropriate to focus on the development of their beliefs about mathematics as a discipline. By setting the development of students' mathematics-related beliefs as an explicit, educational goal, the school might provide the "external influence" that has the potential to change or prevent the self-reinforcing effect of negative beliefs.

A change in one dimension of the belief system will reflect in the other dimensions. Developing the students' beliefs about the beauty, the relevance and the importance of mathematics has the potential to enhance the students' motivation for learning mathematics and to change the focus in mathematics education from results and performance to process and application. Furthermore, it can connect the field of mathematics to other fields and subjects and thus emphasize the interdisciplinary possibilities, i.e. by showing how to use mathematics for problem solving in other subjects or by applying mathematical thinking to other contexts.

Summary and conclusion

The categorization of students' mathematics-related belief system explains how the four dimensions are intertwined and influence each other. This means that a focus on developing one of the belief dimensions will have an effect on the whole belief system. When negative external influences shape the mathematics-related beliefs, it means that it affects not only the beliefs about the context of the influences, but also about what mathematics is, what it is used for, and what is considered appropriate mathematical behaviour. And it affects the self-image as a mathematics learner.

The character of a person's beliefs plays an important role in the person's learning behaviour. As shown in fig. 4, negative beliefs can lead to degressive learning behaviour, resulting in poor mathematical competencies and performance, which subsequently cause more negative experiences. As a consequence, the negative beliefs about mathematics become self-reinforcing.

Awareness of the development of students' mathematics-related beliefs in the most significant external influence – namely the school's mathematics education – has the potential to prevent this self-reinforcing effect of negative beliefs. Focusing on the students' beliefs from the beginning of their education is essential to prevent the beliefs from turning negative.

Particularly beliefs about mathematics as a discipline have the potential to emphasize the importance and relevance of mathematics in the world. These are important motivational factors in learning mathematics. Furthermore: a focus on developing students' beliefs about mathematics as a discipline in the school's mathematics education may also increase the students' awareness of mathematical perspectives in other subjects and disciplines. Deliberately focusing on the students' beliefs about mathematics can both widen their perspective of what mathematics is *and* broaden their horizon and their possibilities in other subjects by giving them the ability to apply mathematical thinking in different contexts.

In short: In order to show students the relevance and the application of mathematics, we need to connect it to other subjects or disciplines and to 'the real world'. This can be done by focusing on their beliefs about what mathematics is – especially their beliefs about mathematics as a discipline.

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